King's College London

Topology

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I cannot stress enough that these notes aren't a substitute for going to the lectures.
The errors in here are my own. Please point any out to me at khallilbenyattou@hotmail.co.uk!
I'd greatly appreciate it ♥

If I highlight the background of something, it either seems important to know, is an exercise or something that Igor stressed in the lectures e.g. topological groups aren't examinable!

I'd like to offer a warm thanks to Wenkui and Joshua for their notes on the three lectures I missed during term. I also owe a debt of gratitude to my friends Aaron Suri and Christopher Nguyen for their continued suggestions on how to improve this set of notes. I can't forget the lecturer Igor himself for answering my queries about examples that were discussed in his lectures. Finally, this couldn't have been done without the support of my family (simultaneously driving me insane, as most families do).

I'd also like to thank the academy. Just kidding.

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Chapter 1

Topological Spaces and Continuous Functions

Preliminaries

A little set theory goes a long way... apparently.

It's possible to define the Cartesian product of an arbitrary indexed family of sets. If I is any index set and $\{X_i\}_{i\in I}$ is a family of sets, then their Cartesian product is defined as

$$\prod_{i\in I} X_i = \left\{ f\colon I \to \bigcup_{i\in I} X_i \colon \forall i \in I,\ f(i) \in X_i \right\}.$$

There are two useful expressions for working with complements and unions of sets. These are called De Morgan's laws. Given a family of subsets $\{A_i\}_{i\in I}$ of a set X,

$$X \setminus \bigcup_{i\in I} A_i = \bigcap_{i\in I} (X \setminus A_i) \qquad X \setminus \bigcap_{i\in I} A_i = \bigcup_{i\in I} (X \setminus A_i)\,.$$

A **relation** on a set A is a subset R of the Cartesian product $A \times A$. If R is a relation on A, we write xRy to mean the same thing as $(x, y) \in R$ and read it as "x is in the relation R to y". We'll consider a special kind of relation called an equivalence relation.

An **equivalence relation** on a set X is a relation $\sim$ on X having the following three properties:

(Reflexivity) For every $x \in X$: $x \sim x$,

(Symmetry) if $x \sim y$, then $y \sim x$,

(Transitivity) and if $x \sim y$ and $y \sim z$, then $x \sim z$.

Given an equivalence relation $\sim$ on X and an element $x \in X$, we define a special subset of X, called the **equivalence class determined by** x, given by $[x] = \{y \in X \colon y \sim x\}$. These form a partition of the set X. More formally, a **partition** of a set X is a collection of disjoint subsets of X whose union is the whole of X.

1.1 Topological Spaces

A topology is informally a structure on a set that allows us to discuss its properties that are preserved under continuous deformation. To motivate the notion of a topology, we begin by discussing facts about the real line. Recall that $A \subseteq \mathbb{R}$ is called open if for every $x \in A$, there exists an $\varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq A$. A set $C \subseteq \mathbb{R}$ is called closed if for every sequence of elements $\{x_n\}_{n\in\mathbb{N}}$ in C, ($x_n \to x$ as $n \to \infty$) $\implies x \in C$.

Proposition 1.1. Let $\mathcal{T} = \{A \subseteq \mathbb{R} \colon A \text{ is open in the sense defined above}\}$. Then $\mathcal{T}$ satisfies the following properties:

- $\emptyset \in \mathcal{T} \ni \mathbb{R}$,
- $\mathcal{T}$ is closed under arbitrary unions and,
- $\mathcal{T}$ is closed under finite intersections.

Proof.

- The empty set vacuously satisfies the criterion for a set to be open.
  Any open set about any point $x \in \mathbb{R}$ is contained in $\mathbb{R}$ so $\mathbb{R} \in \mathcal{T}$.
- For the second property, let $\{U_i\}_{i\in I}$ be a collection of elements of $\mathcal{T}$ and consider their union $U = \bigcup_{i\in I} U_i$. Let $x \in U$. Then $\exists i_0 \in I$ with $x \in U_{i_0}$-open so $\exists r > 0$ such that $x \in (x-r, x+r) \subseteq U_{i_0} \subseteq U$. As x is arbitrary, U is open i.e. $U \in \mathcal{T}$.
- For the final property, consider any finite intersection $V = \bigcap_{i=1}^n U_i$ of a collection $\{U_i\}_{i=1}^n \subseteq \mathcal{T}$ and let $x \in V$. For all $i = 1, \dots, n$, $x \in U_i$-open so $\exists \varepsilon_i > 0$ such that $(x-\varepsilon_i, x+\varepsilon_i) \subseteq U_i \subseteq V$. Take $\varepsilon = \min_{1\leqslant i\leqslant n}\{\varepsilon_i\} > 0$. Then for all $1 \leqslant i \leqslant n$,
  $$(x-\varepsilon, x+\varepsilon) \subseteq (x-\varepsilon_i, x+\varepsilon_i) \subseteq U_i \subseteq V$$
  so V is open. □

Definition 1.2. Let X be an arbitrary set. A collection $\mathcal{T} \subseteq \mathcal{P}(X)$ that satisfies the following conditions is called a **topology** on X:

1) X and $\emptyset \in \mathcal{T}$,
2) $\mathcal{T}$ is closed under arbitrary unions and,
3) $\mathcal{T}$ is closed under finite intersections.

Given an element $A \in \mathcal{T}$, we say A is open (relatively to $\mathcal{T}$). We say that $\mathcal{T}_0$ is the standard topology on $\mathbb{R}$ where $A \subseteq \mathbb{R}$ is open in the usual sense. The complement of an open set $A \in \mathcal{T}$, denoted $X \setminus A$, is called closed.

e.g. Let X be an arbitrary set. Then we can define the following two topologies on X: the discrete topology $\mathcal{T} = \mathcal{P}(X) = 2^X$ and the trivial (or indiscrete) topology $\mathcal{T} = \{\emptyset, X\}$.

e.g. Consider $X = \mathbb{R}$ and $\mathcal{T} = \{(a, b)\colon a < b\}$. Then $\mathcal{T}$ satisfies the first and third axioms of a topology but fails the second. Indeed, consider $U_1 = (1, 2), U_2 = (3, 4) \in \mathcal{T}$ and observe that $(U_1 \cup U_2) \notin \mathcal{T}$.

Proposition 1.3. Let X be an arbitrary set and $\mathcal{T}_f = \{U \subseteq X\colon X \setminus U \text{ is finite}\} \cup \{\emptyset\}$. Then $\mathcal{T}_f$ is a topology on X and is called the finite complement topology on X.

Proof.

- $\emptyset \in \mathcal{T}_f$ by definition and $X \setminus X = \emptyset$ which is certainly finite so $X \in \mathcal{T}_f$.
- Let $\{U_i\}_{i\in I} \subseteq \mathcal{T}_f$ be arbitrary and consider $U = \bigcup_{i\in I} U_i$. Then for all $i \in I$, $X \setminus U_i$ is finite. By De Morgan's law, for any $i \in I$

$$X \setminus U \;=\; X \setminus \bigcup_{i\in I} U_i \;=\; \bigcap_{i\in I} (X \setminus U_i) \;\subseteq\; X \setminus U_i$$

  Therefore $X \setminus U$ is finite so $U \in \mathcal{T}_f$.
- Let $\{U_i\}_{i=1}^n \subseteq \mathcal{T}_f$ and consider $V = \bigcap_{i=1}^n U_i$. Each complement $X \setminus U_i$ is finite and by De Morgan's law,

$$X \setminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \setminus U_i)$$

  the right hand side of which is a finite union of finite sets, and therefore finite so $V \in \mathcal{T}_f$. □

Remark The closed sets in X are those that are finite and X. If X is a finite set, $\mathcal{T}_f$ coincides with the discrete topology.

Definition 1.4. Let X be an arbitrary set and suppose that $\mathcal{T}, \mathcal{T}'$ are two topologies on X. We say that $\mathcal{T}$ is **finer** than $\mathcal{T}'$ if $\mathcal{T}' \subseteq \mathcal{T}$ (equivalently, $\mathcal{T}'$ is **coarser** than $\mathcal{T}$). If either $\mathcal{T}$ is finer than $\mathcal{T}'$ or $\mathcal{T}'$ is finer than $\mathcal{T}$, then we say that $\mathcal{T}, \mathcal{T}'$ are **comparable**.

Proposition 1.5. Consider $X = \mathbb{R}$, let $\mathcal{T}$ be the standard topology on $\mathbb{R}$ and $\mathcal{T}_f$ be the finite complement topology on $\mathbb{R}$. Then $\mathcal{T}$ is finer than $\mathcal{T}_f$.

Proof. Let $U \in \mathcal{T}_f$. Then either $U = \emptyset$ or $\mathbb{R} \setminus F$ where $F = (\bigcup_{k\in K} x_k)$ and K is a finite indexing set. If $U = \emptyset$, then $U \in \mathcal{T}$ so suppose otherwise. For every $x \in \mathbb{R} \setminus F$, $\exists r > 0$ such that $(x - r, x + r) \subseteq \mathbb{R} \setminus F$. Therefore $\mathbb{R} \setminus F \in \mathcal{T}$. Indeed, $\mathcal{T}$ is strictly finer than $\mathcal{T}_f$ as $(0, 1) \in \mathcal{T} \setminus \mathcal{T}_f$. (If F is an empty union, $U = \mathbb{R} \setminus \emptyset$ and $X = \mathbb{R}$ is always in $\mathcal{T}$.) □

1.2 Topological Bases

Definition 1.6. Let X be a set. A **topology basis** on X is a collection of subsets $\mathcal{B} \subseteq \mathcal{P}(X)$ satsfying:

(i) $\forall x \in X.\ \exists B \in \mathcal{B}$ s.t. $x \in B$

(ii) $\forall B_1, B_2 \in \mathcal{B},\ \forall x \in (B_1 \cap B_2).\ \exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subseteq (B_1 \cap B_2)$

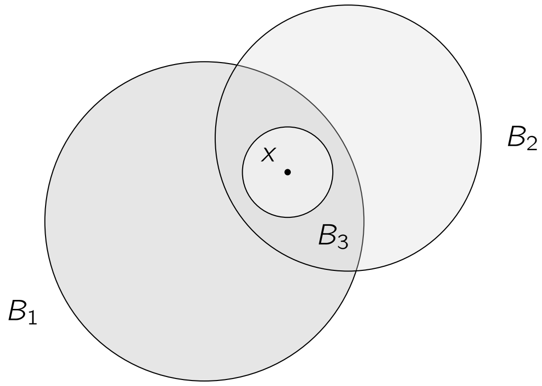


Figure 1.1: An illustration of the second condition if we let $\mathcal{B}$ be the collection of all discs in the plane.

Remark The finite intersection property of a topology is stronger than the second axiom for a topological basis.

Proposition 1.7. Let $\mathcal{B}$ be a topological basis. Then $\mathcal{B}$ generates the following topology $\mathcal{T}$ on X:
For $U \subseteq X$, $U \in \mathcal{T}$ if $\forall x \in U,\ \exists B \in \mathcal{B}$ s.t. $x \in B \subseteq U$.

Proof.

- The empty set satisfies the condition trivially.
  For any $x \in X$, $\exists B \in \mathcal{B}$ such that $x \in B \subseteq X$ so $X \in \mathcal{T}$.

- Let $\{U_i\}_{i \in I} \subseteq \mathcal{T}$. Consider their union $U = \bigcup_{i \in I} U_i$ and let $x \in U$. Therefore, $\exists i_0 \in I$ such that $x \in U_{i_0}$ and because $U_{i_0} \in \mathcal{T}$, $\exists B \in \mathcal{B}$ such that $x \in B \subseteq U_{i_0} \subseteq U \implies U \in \mathcal{T}$.

- Finally, we prove by induction that given $\{U_i\}_{i=1}^n \subseteq \mathcal{T}$, their intersection is also in $\mathcal{T}$. Let $U_1, U_2 \in \mathcal{T}$ and $U = U_1 \cap U_2$. Let $x \in U$. As $x \in U_1 \in \mathcal{T}$, $\exists B_1 \in \mathcal{B}$ s.t. $x \in B_1 \subseteq U_1$. Analogously, $\exists B_2 \in \mathcal{B}$ s.t. $x \in B_2 \subseteq U_2$. Therefore, $x \in B_1 \cap B_2 \subseteq U_1 \cap U_2$ and by the second basis axiom, $\exists B_3$ s.t. $x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$ so $U \in \mathcal{T}$. Assuming that the statement is true for $U_1 \cap \cdots \cap U_{n-1}$, by the result we've shown for two sets, $(U_1 \cap \cdots \cap U_{n-1}) \cap U_n \in \mathcal{T}$.

Therefore, $\mathcal{T}$ is a topology. $\square$

Proposition 1.8. The collection $\mathcal{B}_\ell = \{[a, b) : -\infty < a < b < \infty\}$ is a basis for a topology on $\mathbb{R}$ and this topology is called the **lower limit topology** $\mathcal{T}_\ell$.

Proof. Exercise. □

Lemma 1 Let $\mathcal{B}$ be a topological basis on X and $\mathcal{T}$ be the topology generated by $\mathcal{B}$. Then

$$\mathcal{T} = \left\{ \bigcup_{i \in I} B_i : \{B_i\}_{i \in I} \subseteq \mathcal{B} \right\}.$$

Proof. Denote the right hand side by $\mathcal{T}'$. Let $\bigcup_{i \in I} B_i \in \mathcal{T}'$. Then $\forall i \in I,\ B_i \in \mathcal{T} \implies \bigcup_{i \in I} B_i \in \mathcal{T}$. Let $U \in \mathcal{T}$. Then for all $x \in U$, $\exists B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U \implies U = \bigcup_{x \in U} B_x$. □

Lemma 2 Let $(X, \mathcal{T})$ be a topological space and $\mathcal{C}$ be a collection of open sets in X. If

$$\forall U \in \mathcal{T},\ \forall x \in U\ \exists C \in \mathcal{C}.\ x \in C \subseteq U, \qquad (\star)$$

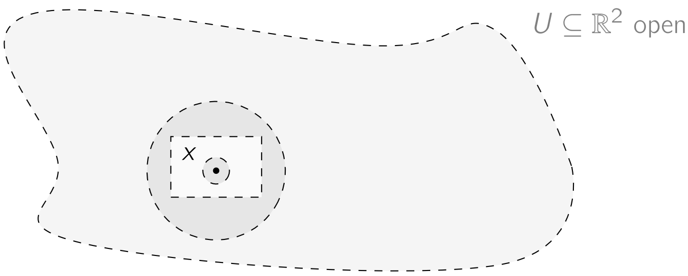
then $\mathcal{C}$ is a topology basis that generates $\mathcal{T}$.

Proof. Firstly, we show that $\mathcal{C}$ is a topological basis. Let $x \in X$ and choose $U = X$. By $(\star)$, $\exists C \in \mathcal{C}$ such that $x \in C \subseteq U = X$. Consider $C_1, C_2 \in \mathcal{C}$, let $U = C_1 \cap C_2 \in \mathcal{T}$ and let $x \in U$. By $(\star)$, $\exists C_3 \in \mathcal{C}$ s.t. $x \in C_3 \subseteq U = C_1 \cap C_2$.

Let $\mathcal{T}'$ be the topology generated by $\mathcal{C}$. Let $U \in \mathcal{T}'$. By lemma 1, we may write it as a union $U = \bigcup_{i \in I} C_i$ where $\{C_i\}_{i \in I} \subseteq \mathcal{C}$ is a collection of open sets. Therefore, $U \in \mathcal{T}$ and $\mathcal{T}' \subseteq \mathcal{T}$. Let $U \in \mathcal{T}$. We need to show that for all $x \in U$, $\exists C \in \mathcal{C}$ such that $x \in C \subseteq U$. This is exactly what we assume in $(*)$ so $U \in \mathcal{T}'$ and so $\mathcal{T} \subseteq \mathcal{T}'$. □

e.g. Consider $X = \mathbb{R}$ equipped with the standard topology and $\mathcal{C} = \{(a, b) : a < b\}$. For any $x \in (a, b) \subseteq \mathbb{R}$, consider (for sufficiently small ε) $C = (x - \varepsilon, x + \varepsilon) \subseteq (a, b)$. $C \in \mathcal{C}$ so using lemma 2, $\mathcal{C}$ is a basis for the standard topology on $\mathbb{R}$.

e.g. Consider $X = \mathbb{R}^2$ equipped with the standard topology. Both $\{(a, b) \times (c, d) \colon a < b \text{ and } c < d\}$ and $\{\mathbb{B}(x, r) \colon x \in \mathbb{R}^2,\ r > 0\}$ are topological bases for the standard topology on $\mathbb{R}^2$. For an intuitive illustration as to why both of these generate the same topology, we can nest open balls and rectangles within one another:



Lemma 3 Suppose that $\mathcal{B}$ and $\mathcal{B}'$ are topological bases on a set X and they generate the topologies $\mathcal{T}, \mathcal{T}'$ respectively. Then TFAE:

1) $\mathcal{T}'$ is finer than $\mathcal{T}$

2) $\forall B \in \mathcal{B}$, $\forall x \in B\ \exists B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$.

Proof. (2) $\implies$ (1). Let $U \in \mathcal{T}$. We wish to show that U is also an element of $\mathcal{T}'$. As $\mathcal{B}$ generates $\mathcal{T}$, for every $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq U$. By the second condition, $\forall x \in U\ \exists B' \in \mathcal{B}'$ s.t. $x \in B' \subseteq B$. Then $x \in B' \subseteq U$ so $U \in \mathcal{T}'$.

(1) $\implies$ (2). The definition of the topology $\mathcal{T}'$ is that $\forall U \in \mathcal{T}'\ \forall x \in U$, $\exists B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$. Take a basic open neighbourhood $B \in \mathcal{B}$. Then $B \in \mathcal{T}$ and because $\mathcal{T}'$ is finer than $\mathcal{T}$, this means that $B \in \mathcal{T}'$. Therefore, for all $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. □

1.3 The Product Topology ($n = 2$)

We begin with two topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and want to define a topology on $X \times Y$. A natural suggestion is to consider the collection $\{U \times V : U \in \mathcal{T}_X,\ V \in \mathcal{T}_Y\}$.

e.g. Consider $X = Y = \mathbb{R}$ and take the following sets:

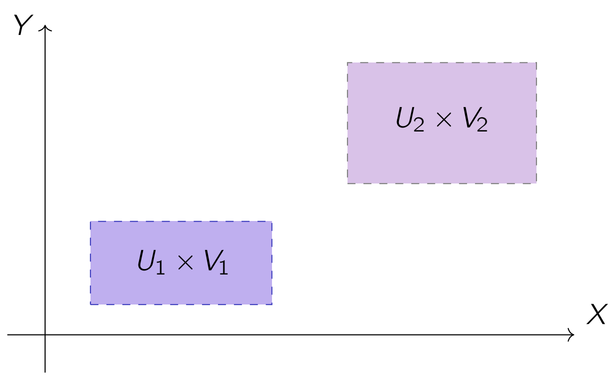


Figure 1.2: Two elements of the collection whose union doesn't belong to the collection.

This collection isn't closed under unions and so doesn't define a topology on X. Fortunately this collection does define a basis for a topology.

Definition 1.9. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. The **product topology** on $X \times Y$ is the topology generated by the collection $\mathcal{B} = \{U \times V : U \subseteq X \text{ open}, V \subseteq Y \text{ open}\}$.

Is it indeed a topological basis? To show that the collection covers $X \times Y$, note that every $(x, y) \in X \times Y$ and X and Y are open in themselves. For the second axiom of a topology basis, we'll prove the stronger finite intersection property (of a topology). Let $(U \times V)$ and $(U' \times V') \in \mathcal{B}$. Then $(U \times V) \cap (U' \times V') = \underbrace{\left(U \cap U'\right)}_{\text{open in } X} \times \underbrace{\left(V \cap V'\right)}_{\text{open in } Y} \in \mathcal{B}$.

e.g. Consider $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ where $\mathbb{R}^2$ and $\mathbb{R}$ are equipped with their respective standard topologies and the Cartesian product is endowed with the product topology. Do both of these topologies coincide? A more general statement is below and will serve to definitively answer this problem.

Proposition 1.10. Let X and Y be two topological spaces generated by the topology bases $\mathcal{B}_X$ and $\mathcal{B}_Y$ respectively. Then the collection $\mathcal{B} = \{B \times C : B \in \mathcal{B}_X, C \in \mathcal{B}_Y\}$ generates the product topology.

Proof. Given an open set $W \subseteq X \times Y$ and a point $(x, y) \in W$, by the definition of the product topology there is a basis element $U \times V$ such that $(x, y) \in U \times V \subseteq W$. As $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for X and Y respectively, $\exists B \in \mathcal{B}_X$ and $C \in \mathcal{B}_Y$ such that $x \in B \subseteq U$ and $y \in C \subseteq V$. Then $(x, y) \in B \times C \subseteq W$ so $\mathcal{B}$ meets the criterion of lemma 2. Therefore, $\mathcal{B}$ is a basis for $X \times Y$. □

e.g. Consider the projection maps π_1 and π_2 defined by

$$\begin{aligned} \pi_1 \colon X \times Y &\longrightarrow X \\ (x, y) &\longmapsto x \end{aligned} \qquad \begin{aligned} \pi_2 \colon X \times Y &\longrightarrow Y \\ (x, y) &\longmapsto y. \end{aligned}$$

The product topology is the coarsest topology that makes the projection maps continuous (a property of maps between topological spaces that we shall define later).

1.4 Subspace Topology

Let Y be a subset of a topological space X. A natural question that arises concerns how we can endow Y with a topology. To do so, we use the existing topology on X:

Definition 1.11. Let X be a topological space and $Y \subseteq X$. The **subspace topology** on Y is defined to be $\mathcal{T}_Y = \{U \cap Y : U \subseteq X \text{ is open}\}$.

Proof. Denote the topology on X by $\mathcal{T}$. $\emptyset$ is open in X so $\emptyset \cap Y = \emptyset \in \mathcal{T}_Y$. Also, X is open in itself so $X \cap Y = Y \in \mathcal{T}_Y$. Let $\{A_i\}_{i \in I} \subseteq \mathcal{T}_Y$. Then for each $i \in I$, $\exists U_i$ open in X with $A_i = U_i \cap Y$. Then

$$\bigcup_{i \in I} A_i = \bigcup_{i \in I} (U_i \cap Y) = \underbrace{\left(\bigcup_{i \in I} U_i\right)}_{\in \mathcal{T}} \cap Y$$

so $\mathcal{T}_Y$ is closed under arbitrary unions. For finite intersections, the argument is very similar. □

e.g. Let $X = \mathbb{R}$ and $Y = [-1, 1]$. Are the following sets open in Y?

a) $A = \{x \in \mathbb{R} : \frac{1}{2} < |x| < 1\}$
   A is open in X and is also a subset of Y so is open in Y.

b) $B = \{x \in \mathbb{R} : \frac{1}{2} < |x| \leqslant 1\}$
   Take $a \in \mathbb{R}$ with $|a| > 1$. Then $B = \left((-a, -\frac{1}{2}) \cup (\frac{1}{2}, a)\right) \cap Y$ so B is open in Y with the subspace topology.

c) $C = \{x \in \mathbb{R} : \frac{1}{2} \leqslant |x| \leqslant 1\}$
   We claim that C isn't open in Y. If C were open in Y, it'd be equal to $U \cap Y$ for some U open in X. As U is open and $-1/2 \in U$, there exists an $\varepsilon > 0$ such that $(-1/2 - \varepsilon, -1/2 + \varepsilon) \subseteq U$. However, $(-1/2, -1/2 + \varepsilon) \nsubseteq C = U \cap Y$. These two statements contradict each other. The claim follows.

e.g. Let $X = \mathbb{R}$ and $Y = (-1, 1)$. Is $A = \{x \in \mathbb{R} : \frac{1}{2} \leqslant |x| < 1\}$ open in Y?

Claim A is not open.

Proof. Notice that Y is open. If A were open, there would exist an open subset U of X so that $A = U \cap Y$ which is open in $\mathbb{R}$. However, A isn't open in $\mathbb{R}$ so it can't be open in Y. □

The last example leads us to consider a short lemma (and its converse).

Lemma Suppose that $(X, \mathcal{T})$ is a topological space and Y is a subspace of X. If $U \subseteq Y$ is open in Y and Y is open in X, then U is open in X.

Proof. If U is open in Y, then there exists a $V \subseteq X$ open in X such that $U = V \cap Y$. Since Y is open in X, U is open in X as the intersection of two sets that are open in X. □

Remark The converse states that if ($\forall A \subseteq Y$ open $\implies$ $A \subseteq X$ open), then $Y \subseteq X$ is open in X.

e.g. For $S^1 \hookrightarrow \mathbb{R}^2$, open sets in S^1 "look like" open intervals in $\mathbb{R}$.

e.g. The torus $T \subseteq \mathbb{R}^3$ inherits the product topology from its Cartesian product representation, $S^1 \times S^1$.

e.g. Let $Y = [0,1]^2 \subseteq \mathbb{R}^2$ and consider the set $\mathcal{D} = \left\{x \in [0,1]^2 \colon \|x - (1,1)\| < \frac{1}{\sqrt{5}}\right\} = \mathbb{B}_{\frac{1}{\sqrt{5}}}((1,1)) \cap Y$.

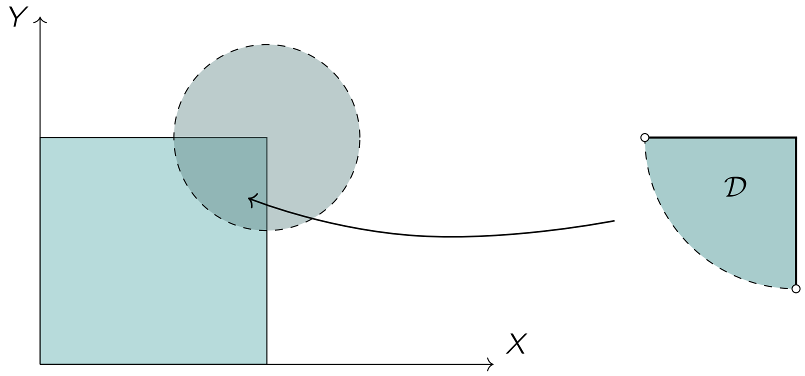


Figure 1.3: An illustration of $\mathcal{D}$ which is open in Y. Note that we include neither the arc nor its endpoints.

Lemma 4 Let $Y \subseteq X$ be a subspace (i.e. equipped with the subspace topology) and $\mathcal{B}$ be a topological basis on $(X, \mathcal{T}_X)$. Then $\mathcal{B}_Y = \{Y \cap B \colon B \in \mathcal{B}\}$ is a basis for a topology on Y. Indeed, this topology is the subspace topology on Y.

Proof. Firstly, we show that this is a topological basis. Let $y \in Y$ so $\exists B \in \mathcal{B}$ such that $y \in B \subseteq X$ and so $y \in B \cap Y \subseteq X$. This means that $\mathcal{B}_Y$ is a cover of Y. For the second property, suppose that $y \in (B_1 \cap Y) \cap (B_2 \cap Y) = (B_1 \cap B_2) \cap Y$ where $B_1, B_2 \in \mathcal{B}$. As $\mathcal{B}$ is a basis for a topology on X, we have that $\exists B_3 \in \mathcal{B}$ such that $y \in B_3 \subseteq (B_1 \cap B_2)$. Consequently, $y \in (B_3 \cap Y) \subseteq (B_1 \cap B_2) \cap Y$. Therefore $\mathcal{B}_Y$ is a basis for a topology on Y, call it $\mathcal{T}$.

The subspace topology on Y is $\mathcal{T}_Y = \{U \cap Y \colon U \subseteq X \text{ is open}\}$ and by lemma 1,

$$\mathcal{T} = \left\{\bigcup_{i \in I} B_i : \{B_i\}_{i \in I} \subseteq \mathcal{B}_Y\right\}.$$

We'll show that these two are the same. Let $U \in \mathcal{T}$. Then there exists a collection $\{B_i\}_{i \in I} \subseteq \mathcal{B}$ such that

$$U = \bigcup_{i \in I} (B_i \cap Y) = \underbrace{\left(\bigcup_{i \in I} B_i\right)}_{\in \mathcal{T}_X} \cap Y \implies U \in \mathcal{T}_Y.$$

For the reverse inclusion, let $U \in \mathcal{T}_Y$. Then there exists some V that is open in X with $U = V \cap Y$. Since V is open in X, it can be written as the union of a collection of elements of $\mathcal{B}$ and so $U \in \mathcal{T}$. Thus, $\mathcal{B}_Y$ is a basis for the subspace topology on Y. □

When it comes to embedding spaces in their ambient spaces, a small problem arises; Let $X = \mathbb{R}^2$ and $x_0 \in \mathbb{R}$. We can embed $\mathbb{R} \hookrightarrow \mathbb{R}^2$ via the map $x \mapsto (x_0, x)$. Now the question is which topology on $\mathbb{R}$ is induced by our embedding into $\mathbb{R}^2$, the product topology or subspace topology? In this case, it seems intuitive that we can associate the product topology on $X \times \{x_0\}$ (where $\{x_0\}$ is endowed with the trivial topology) with the topology on X via the map $x \mapsto (x, x_0)$. More generally, we have a theorem to talk about this.

Theorem 1.12 Let X, Y be topological spaces and $A \subseteq X$, $B \subseteq Y$ be subspaces. Let $\mathcal{T}_{A\times B}$ be the product topology on $A \times B$ and $\widetilde{\mathcal{T}}_{A\times B}$ be the subspace topology on $A \times B \subseteq X \times Y$. Then $\mathcal{T}_{A\times B} = \widetilde{\mathcal{T}}_{A\times B}$.

Proof. It's easier to work with topological bases.
Let $\mathcal{C}_X, \mathcal{C}_Y$ be any topological bases on X, Y respectively.

By proposition 1.10, $\{C \times C' \colon C \in \mathcal{C}_X,\ C' \in \mathcal{C}_Y\}$ is a topological basis for the product topology on $X \times Y$. By lemma 4, $\{(C \times C') \cap (A \times B) : C \in \mathcal{C}_X,\ C' \in \mathcal{C}_Y\} =: \widetilde{\mathcal{B}}_{A\times B}$ is a topology basis for $\widetilde{\mathcal{T}}_{A\times B}$.

By lemma 4, $\{C \cap A \colon C \in \mathcal{C}_X\}$ and $\{C' \cap A \colon C' \in \mathcal{C}_Y\}$ are topological bases for the subspace topologies on A and B respectively. By proposition 1.10, $\{(C \cap A) \times (C' \cap B) \colon C \in \mathcal{C}_X,\ C' \in \mathcal{C}_Y\} =: \mathcal{B}_{A\times B}$ generates the topology $\mathcal{T}_{A\times B}$.

Set theoretically, both topological bases coincide i.e. $\widetilde{\mathcal{B}}_{A\times B} = \mathcal{B}_{A\times B}$ so they generate the same topology. □

Remark Note that finding two topology bases that differ doesn't necessarily imply that they generate different topologies.

1.5 Closed Sets and Limit Points

Definition 1.13. A set $A \subseteq X$ (where X is a topological space) is **closed** in X if $X \setminus A$ is open in X. A set can be closed and open at the same time. Such sets are called clopen. The empty set and X are always clopen subsets of a topological space X.

e.g. Consider $Y = (0,1) \cup (2,3) \subseteq \mathbb{R} = X$ and let $A = (0,1)$. A is clearly open in the subspace topology. It's closed in Y because $Y \setminus A = (2,3)$ is open. Therefore, A is clopen in Y.

Definition 1.14. Let $Y \subseteq X$ be a subspace and $A \subseteq Y$. We say that A is **closed in** Y if $Y \setminus A$ is open in the subspace topology on Y.

It turns out that we can characterise a topology entirely by closed sets. The very definition of a closed set as the complement of an open one allows for such a characterisation.

Theorem 1.15 Let $(X, \mathcal{T})$ be a topological space. The following conditions hold:

- $\emptyset, X$ are closed in X.
- For any collection $\{C_i\}_{i\in I}$ of closed sets in X, their arbitrary intersection $\bigcap_{i\in I} C_i$ is still closed.
- Given any finite family of closed sets $\{C_i\}_{i=1}^n$, their union $\bigcup_{i=1}^n C_i$ is still closed.

Proof.

- The empty set and the whole space are both open and complements of each other so must also be closed in X.
- Let $\{C_i\}_{i\in I}$ be a collection of closed sets in X. This means that for each $i \in I$, $X \setminus C_i$ is an open set in X. As $\mathcal{T}$ is a topology, it's closed under arbitrary unions. By De Morgan's law, we see that the set below is an open set so the arbitrary intersection must be closed.

$$X \setminus \bigcap_{i\in I} C_i = \bigcup_{i\in I}(X \setminus C_i)$$

- Let $\{C_i\}_{i=1}^n$ be a finite collection of closed sets in X. For each $1 \leqslant i \leqslant n$, $X \setminus C_i$ is an open set. Using that $\mathcal{T}$ is closed under finite intersections and De Morgan's law again tells us that the complement

$$X \setminus \bigcup_{i=1}^n C_i = \bigcap_{i=1}^n (X \setminus C_i)$$

is an open set so the finite union of the C_i must be closed.

□

Theorem 1.16 Let $Y \subseteq X$ be a subspace of X. Then $A \subseteq Y$ is closed in $Y \iff \exists C \subseteq X$ closed s.t. $C \cap Y = A$.

Proof. Assume that $A = C \cap Y$ where C is closed in X. Then $X \setminus C$ is open in X. This means that $(X \setminus C) \cap Y$ is open in Y in the subspace topology. Notice that $(X \setminus C) \cap Y = Y \setminus A$. Hence $Y \setminus A$ is open in Y, so that A is closed in Y.

Conversely, assume that A is closed in Y. Then $Y \setminus A$ is open in Y so there exists an open subset U of X s.t. $Y \setminus A = U \cap Y$. Since U is open in X, $X \setminus U$ is closed in X. Notice that $A = Y \cap (X \setminus U)$ and the claim follows. □

Remark Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

1.5.1 Closure and Interior

Recall that for a subset $A \subseteq \mathbb{R}$, we define its interior as $\text{int}(A) = \{x \in A \colon \exists \varepsilon > 0 \text{ s.t. } (x-\varepsilon, x+\varepsilon) \subseteq A\}$. More generally, we have the following definition for a topological space X.

Definition 1.17. If $A \subseteq X$, then the **interior** of A is $\text{int}(A) = \bigcup_{\substack{U\subseteq A \\ \text{open}}} U.$

Remark The interior is the largest open subset of A in the sense that if $V \subseteq A$ is any open subset, then $V \subseteq \text{int}(A) \subseteq A$. The interior is always open because it's the union of a collection of open sets. Note that $\text{int}(A) = A \iff A$ is open.

Recall that for a subset $A \subseteq \mathbb{R}$, we define its closure $\overline{A}$ by the set of limits of sequences of elements of A. We also know that A is closed $\iff A = \overline{A}$.

Definition 1.18. If $A \subseteq X$, then the **closure** of A is $\overline{A} = \bigcap\limits_{\substack{C \supseteq A \\ \text{closed}}} C$.

Remark The closure of a set is the smallest closed set that contains A in the sense that if C is any closed set containing A, then $C \supseteq \overline{A} \supseteq A$. Also, $A = \overline{A} \iff A$ is closed.

Definition 1.19. The **boundary of** A is $\partial A := \overline{A} \setminus \text{int}(A)$.

To check that a point $x \in A$ belongs to the interior of A, it's enough to check that there exists an open set U in X with $x \in U \subseteq A$. For closed sets, life isn't so easy. Luckily, we can formulate the closure in terms of the interior and vice versa.

Proposition 1.20. The following are equivalent:

(i) $\overline{A} = X \setminus \text{int}(X \setminus A)$

(ii) $\text{int}(A) = X \setminus \overline{(X \setminus A)}$

Proof.

$$X \setminus \text{int}(X \setminus A) \;=\; X \setminus \left(\bigcup_{\substack{U \subseteq (X \setminus A) \\ \text{open}}} U \right) \;=\; \bigcap_{\substack{U \subseteq (X \setminus A) \\ \text{open}}} (X \setminus U) \overset{(\dagger)}{=} \bigcap_{\substack{C \supseteq A \\ \text{closed}}} C \;=\; \overline{A}$$

At $(\dagger)$, we rewrite $X \setminus U = C$ and note that C is closed as its complement is open. Re-writing it as such, we have the form of the closure of A.

As for why they're equivalent, you can get (ii) from (i) by replacing A with $X \setminus A$

$$\begin{aligned} \overline{X \setminus A} &= X \setminus \text{int}(X \setminus (X \setminus A)) \\ &= X \setminus \text{int}(A) \end{aligned}$$

and taking complements $X \setminus (...)$ on both sides. Working backwards, we easily get (i) from (ii). □

Theorem 1.21 Let X be a topological space, $Y \subseteq X$ be a subspace of X and $A \subseteq Y$. Denote the closure of A in X by $\overline{A}$. The closure of A in Y is given by $\overline{A} \cap Y$.

Proof. Let B denote the closure of A in Y. The set $\overline{A}$ is closed in X so $\overline{A} \cap Y$ is closed in Y (by theorem 1.16). Since $\overline{A} \cap Y$ contains A, and since by definition B is the intersection of all closed subsets of Y containing A, we must have $B \subseteq \left(\overline{A} \cap Y\right)$.

On the other hand, we know that B is closed in Y so $\exists C$ closed in X such that $B = C \cap Y$. Then C is a closed set in X containing A. As $\overline{A}$ is the intersection of all such closed sets, $\overline{A} \subseteq C$. Then $\left(\overline{A} \cap Y\right) \subseteq (C \cap Y) = B$. □

e.g. Let X be an arbitrary set and $\mathcal{T}$ be the discrete topology on X. Take $A \subseteq X$. Then $\overline{A} = A$, $\text{int}(A) = A$ and $\partial A = A \setminus A = \emptyset$.

Proposition 1.22. $\partial A = \emptyset \iff A$ is clopen in X.

Proof. A is both open and closed so $\overline{A} = A = \text{int}(A)$ and so $\partial A = \overline{A} \setminus \text{int}(A) = A \setminus A = \emptyset$. For the converse, suppose that $\partial A = \emptyset = \overline{A} \setminus \text{int}(A)$ and recall that $\text{int}(A) \subseteq A \subseteq \overline{A}$. (This means that there'd be nothing in $\overline{A}$ that you couldn't find in $\text{int}(A)$). □

e.g. Let X be an infinite set, $\mathcal{T}_f$ be the finite complement topology on X and $A \subseteq X$. Then

$$\text{int}(A) = \begin{cases} A, & \text{if } |X \setminus A| < \infty \\ \emptyset, & \text{if } |X \setminus A| = \infty \end{cases} \qquad \overline{A} = \begin{cases} A, & \text{if } |A| < \infty \\ X, & \text{if } |A| = \infty. \end{cases}$$

Using these, we can calculate the boundary of A:

$$\partial A = \begin{cases} X \setminus X = \emptyset, & \text{if } |A| = \infty \text{ and } |X \setminus A| = \infty \\ A \setminus \emptyset = A, & \text{if } |A| < \infty \text{ and } |X \setminus A| = \infty \\ X \setminus A, & \text{if } |A| = \infty \text{ and } |X \setminus A| < \infty \\ \text{☡} & \text{if } |A| < \infty \text{ and } |X \setminus A| < \infty \end{cases}$$

For the last case, having $|A| < \infty$ and $|X \setminus A| < \infty$ occur simultaneously isn't possible as that'd mean the union of two finite sets (A and $X \setminus A$) is an infinite set (X).

Remark Note that if $|X| < \infty$, $\mathcal{T}_f = \mathcal{P}(X)$ so the prior example applies to any $A \subseteq X$.

e.g. Consider $\mathbb{Q} \subsetneq \mathbb{R}$.

Let $r \in \text{int}(\mathbb{Q})$. This occurs if and only if $\exists \varepsilon > 0$ such that $(r - \varepsilon, r + \varepsilon) \subseteq \text{int}(\mathbb{Q})$. However, every neighbourhood of r must contain an irrational point by the completeness of the real line. Therefore, $\text{int}(\mathbb{Q}) = \emptyset$. By definition, $\overline{\mathbb{Q}} = \mathbb{R} \setminus \text{int}(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R} \setminus \emptyset = \mathbb{R}$. Also, $\partial \mathbb{Q} = \overline{\mathbb{Q}} \setminus \text{int}(\mathbb{Q}) = \mathbb{R} \setminus \emptyset = \mathbb{R}$.

e.g. Consider $A = \mathbb{Q}^2 \subseteq \mathbb{R}^2$.

The complement of $\mathbb{Q}^2$ in $\mathbb{R}^2$ is the set of tuples (q_1, q_2) where at least one of q_1 and q_2 is irrational. Every neighbourhood of a point in the complement $\mathbb{R}^2 \setminus \mathbb{Q}^2$ contains a point whose coordinates are both rational so its interior is empty. Thus $\overline{\mathbb{Q}^2} = \mathbb{R}^2 \setminus \text{int}(\mathbb{R}^2 \setminus \mathbb{Q}^2) = \mathbb{R}^2 \setminus \emptyset = \mathbb{R}^2$.

e.g. Consider $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}$

The interior of S is empty. $\overline{S} = \mathbb{R} \setminus \text{int}(\mathbb{R} \setminus S)$. Note that $0 \notin S$ and that

$$\text{int}(\mathbb{R} \setminus S) = (-\infty, 0) \cup \bigcup_{n \in \mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n} \right) \cup (1, \infty) \implies \overline{S} = S \cup \{0\}.$$

Remark It appears as though the procedure of taking the closure of a set involves adding accumulation points. In the prior example, 0 is where the sequence $\{1/n\}_{n\in\mathbb{N}}$ seems to converge.

e.g. Consider $S \subseteq (0,1]$ (equipped with subspace topology induced by $\mathbb{R}$)

Our theorem about closures in subspaces tells us that $\overline{S_Y} = \overline{S} \cap Y = \overline{S} \cap (0,1] = (S \cup \{0\}) \cap (0,1] = S$. Therefore, S is closed in $(0,1]$ but not in $\mathbb{R}$!

e.g. Consider $A = \mathbb{Z}^2 \subseteq \mathbb{R}^2$.

Seeing as $\mathbb{R}^2 \setminus \mathbb{Z}^2$ is an open set, it's equal to its own interior. Accordingly, $\overline{\mathbb{Z}^2} = \mathbb{R}^2 \setminus (\mathbb{R}^2 \setminus \mathbb{Z}^2) = \mathbb{Z}^2$.

Now we'll talk about deciding whether a point is in the closure of a set or not.

Definition 1.23. A **neighbourhood** of a point $x \in X$ is any open set $U \subseteq X$ containing x.

Remark We say that two sets A, B intersect (non-emptily) if $A \cap B \neq \emptyset$.

Theorem 1.24 Let A be a subset of a topological space $(X, \mathcal{T})$. Then

a) For all $x \in X$, $x \in \overline{A} \iff$ every neighbourhood of x intersects A.

b) If $\mathcal{T}$ is generated by $\mathcal{B}$, then $x \in \overline{A} \iff \forall B \in \mathcal{B}$ that's a basic open neighbourhood of x, $B \cap A \neq \emptyset$.

c) $x \in \partial A \iff \forall$ (basic) neighbourhood U of x, U intersects both A and $X \setminus A$.

Proof. Doing this by contrapositive is a lot easier than doing it directly. We'll prove a):

- For the forward direction, suppose that there exists a neighbourhood U of x that doesn't intersect A. We want that $x \notin \overline{A}$. As $x \in U$ and $U \cap A = \emptyset$, we can say that $A \subseteq X \setminus U$ which is closed. By the minimality of the closure of a set, $A \subseteq \overline{A} \subseteq V$ for any closed V in X. Therefore,

$$A \subseteq \overline{A} \subseteq X \setminus U \not\ni x \implies x \notin \overline{A}.$$

- For the reverse direction, suppose that $x \notin \overline{A}$. In particular, $x \notin A$. Now $X \setminus \overline{A}$ is an open set that contains x so is a neighbourhood of x with $U \cap \overline{A} = \emptyset$. Finally, $(U \cap A) \subseteq (U \cap \overline{A})$ so we've found a neighbourhood of x that doesn't intersect A.

□

e.g. Consider $(\alpha, \beta) \subseteq \mathbb{R}$. Note that α and β are the only points in $\mathbb{R}$ for which every neighbourhood containing them intersects (α, β) and $(-\infty, \alpha] \cup [\beta, \infty)$. Therefore, $\partial(\alpha, \beta) = \{\alpha, \beta\}$.

e.g. $A = \mathbb{R} \setminus \mathbb{Q}$. For the same reason as the prior example, $\overline{A} = \partial A = \mathbb{R}$.

e.g. $A = \mathbb{R} \subseteq \mathbb{R} = X$. For the boundary, there aren't any points x in $\mathbb{R}$ whose (basic) open neighbourhoods intersect $\mathbb{R}$ and $\mathbb{R} \setminus A = \emptyset$ so $\partial \mathbb{R} = \emptyset$.

e.g. $A = (a, b) \subseteq \mathbb{R}_u$ where $\mathbb{R}_u$ is the real line equipped with the topology generated by the set $\mathcal{B} = \{(\alpha, \beta] \colon -\infty < \alpha < \beta < \infty\}$. We call the generated topology the upper limit topology, $\mathcal{T}_u$.

Since A can be written as

$$A = \bigcup_{n \in \mathbb{N}} (a, b - 1/n] \,,$$

$\mathrm{int}(A) = A$. As for the closure, $\overline{A} = [a, b]$ because every neighbourhood about a intersects A. The same applies to b but for every point $c > b$, there exists some $\delta > 0$ such that $(c - \delta, c + \delta) \cap (a, b] = \emptyset$. Likewise for any $d < a$.

Remark On $\mathbb{R}$, the upper limit topology $\mathcal{T}_u$ is finer than the standard topology.

1.5.2 Limit Points

If $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, we say x is a limit point of A (written $x \in A'$) if $\exists \{x_n\}_{n \in \mathbb{N}} \subseteq A \setminus \{x\}$ s.t. $x_n \to x$ as $n \to \infty$. An equivalent way of saying this is that $\exists N$ s.t. $n \geqslant N \implies x_n \in (x - \varepsilon, x + \varepsilon) \setminus \{x\}$. If we call this interval U, it seems to suggest a limit point of a set is one that can be approximated by points of the set.

Definition 1.25. Let $A \subseteq X$ where X is a topological space. We say that $x \in X$ is a **limit** (accumulation/cluster) **point of** A if every neighbourhood U of x non-emptily intersects $A \setminus \{x\}$. We denote the set of limit points of A by A'.

e.g. $(0, 1)' = \overline{(0, 1)} = [0, 1]$

e.g. For $S = \left\{\frac{1}{n} \colon n \in \mathbb{N}\right\} \subseteq \mathbb{R}$, we saw that $\overline{S} = S \cup \{0\}$. Note that $S' = \{0\} \subsetneq \overline{S}$.

Remark In particular, $A' \subseteq \overline{A}$.

Theorem 1.26 $\overline{A} = A \cup A'$

Remark This may or may not be a disjoint decomposition. It's disjoint if and only if every point of A is an isolated point (i.e. $\forall x \in A \; \exists U \subseteq X$ that's a neighbourhood of x that only intersects A at $\{x\}$).

Proof. $A' \subseteq \overline{A}$ as points in A' satisfy a stronger condition than those in $\overline{A}$. Namely, intersecting open neighbourhoods with A is a weaker condition than intersecting with $A \setminus \{x\}$.

For the reverse inclusion, let $x \in \overline{A} \setminus A$. If we can show that x necessarily lies in A', we'll have shown the reverse inclusion. $x \in \overline{A}$ means that every neighbourhood U of x intersects A non-emptily i.e. $U \cap A \neq \emptyset$. As $x \notin A$, we can write that $A = A \setminus \{x\}$. This means that $U \cap (A \setminus \{x\}) \neq \emptyset$ and U is arbitrary so $x \in A'$. $\square$

1.5.3 Convergence of Sequences

Recall that if $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, $x_n \to x$ as $n \to \infty$ if $\forall \varepsilon > 0$, $\exists N > 0$ such that $n \geqslant N \implies \|x_n - x\| \leqslant \varepsilon$. This is equivalent to saying that $x_n \in (x - \varepsilon, x + \varepsilon)$.

In this case, we say that all but finitely many elements of the sequence lie in the ε-neighbourhood of the point x.

Definition 1.27. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ be a sequence and $x \in X$. We say that x_n **converges to** x (i.e. $x_n \to x$ as $n \to \infty$) if $\forall U \subseteq X$ neighbourhood of x, $\exists N = N(U)$ such that $n \geqslant N \implies x_n \in U$.

Remark So far, we haven't discussed if the limit of a sequence (if it exists) is unique. As we'll see in the next (somewhat pathological) example, it isn't unique in general.

e.g. Let X be arbitrary and $\mathcal{T}$ be the trivial topology on X. Take any sequence $\{x_n\}$ in X. The only candidate for a neighbourhood of our conjectured limit x is X. For $U = X$, the definition is satisfied for $N = 1$. Therefore, any sequence converges to every element of X.

e.g. $(X, \mathcal{P}(X))$ where $X = \mathbb{R}$ and $x_n = (a, b, c, 1, 1, 1, \dots)$. Does x_n converge to a limit?

Proposition 1.28. Let $(X, \mathcal{P}(X))$ be a topological space and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X. Then x_n converges if and only if it's eventually constant.

Proof. For the forward implication, suppose that $\{x_n\}_{n\in\mathbb{N}}$ converges to some $x \in X$. This means that for every neighbourhood U of x, $\exists N = N(U)$ such that $n \geqslant N \implies x_n \in U$. In particular, take $\{x\} = U$. Then we have that $\exists N$ such that $n \geqslant N \implies x_n \in \{x\}$ i.e $\{x_n\}_{n\in\mathbb{N}}$ is eventually constant.

For the reverse implication, suppose that x_n is eventually constant i.e. $\exists N$ and $\exists x \in X$ such that $n \geqslant N \implies x_n = x$. Since X is equipped with the discrete topology, $\{x\}$ is certainly an example of a neighbourhood of x. Any neighbourhood of x will contain $\{x\}$ so we're done. □

e.g. For $\mathbb{R}$ equipped with the standard topology (or $\mathbb{R}^n$ with the standard topology), we have convergence in the usual sense i.e. $x_n \to x$ as $n \to \infty$ iff $\forall \varepsilon > 0$, $\exists N > 0$ such that $n \geqslant N \implies \|x_n - x\| \leqslant \varepsilon$.

e.g. $X = \mathbb{R}$ with the finite complement topology $\mathcal{T}_f$.

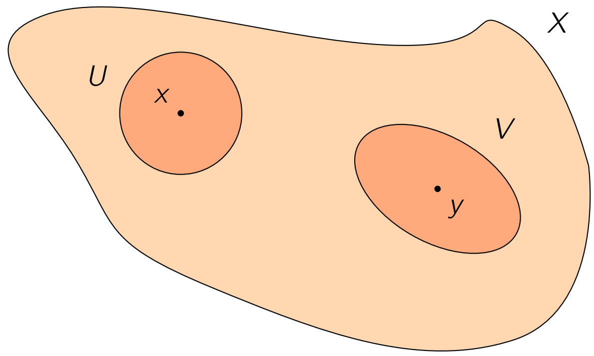
a) Consider $x_n = (-1)^n$. The natural candidates for convergence are -1 and 1. We claim that $x_n \not\to -1$. To disprove the convergence of a sequence to a point, it's sufficient to find a neighbourhood for which the definition doesn't hold. Take $U = \mathbb{R} \setminus \{+1\} \in \mathcal{T}_f$. U is a neighbourhood of -1. However, $1 = x_{2n} \notin U$ for an unbounded collection of n. Therefore the claim holds. A similar argument holds for $x_n \not\to +1$.

b) Consider $x_n = n$. Take $x \in \mathbb{R}$ and let $U = \mathbb{R} \setminus \bigcup_{i=1}^k \{a_i\}$ be a neighbourhood of x. If $n > \max_{1\leqslant i\leqslant k}\{a_i\}$ (note that the max is $< \infty$), then $x_n = n \in \mathbb{R} \setminus \bigcup_{i=1}^k \{a_i\}$. Therefore, $\forall x \in \mathbb{R}$, $x_n \to x$.

Remark In fact, any sequence that doesn't attain a given value infinitely many times has the same property: that it converges to every $x \in \mathbb{R}$. (In the finite complement topology)

Proof. Exercise □

Definition 1.29. A topological space X is **Hausdorff** (or satisfies the T_2 separation axiom) if $\forall x, y \in X$ with $x \neq y$, $\exists U, V$ neighbourhoods of x and y respectively such that $U \cap V = \emptyset$.



Remark The property of a topological space being Hausdorff implies the uniqueness property of the convergence of a sequence in the space (if the limit exists).

Theorem 1.30 Let X be a Hausdorff space and S be a finite subset of X. Then S is closed in X.

Remark This theorem is equivalent to the T_1 separation axiom: that for any two distinct points in the set, x and y, there exists an open neighbourhood U about x that doesn't contain y.

Proof. Let X be Hausdorff and S a finite subset of X. It's sufficient to prove that the singleton set is closed i.e. that its complement in X is open. Denote $U = X \setminus \{x\}$. Therefore for any $y \in U$, there are disjoint open neighbourhoods U_x and V_y containing x and y respectively. Then,

$$U = X \setminus \{x\} = \bigcup_{y \in U} V_y$$

is an open set. □

Theorem 1.31 (Hausdorff $\implies$ uniqueness of limits) Suppose that $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in a Hausdorff space X. Then x_n converges to at most one limit.

Remark The converse statement is not true. However, if a topological space has the property that every sequence converges to at most one limit, then X is T_1.

e.g. The real line equipped with the standard topology is Hausdorff. Therefore, sequences have at most one limit and $\mathbb{R}$ is T_1.

e.g. Let X be arbitrary, $(X, \mathcal{P}(X))$ be a topological space and x and y be distinct elements of X. The singletons $\{x\}$ and $\{y\}$ are of course open and disjoint so satisfy the criterion for a Hausdorff space.

e.g. Let X be an infinite set equipped with the finite complement topology $\mathcal{T}_f$. Consider two distinct points x, y in X. Any two neighbourhoods of x and y respectively must intersect (as their complements are finite). Therefore, X isn't T_2. Is it T_1? We can either show this directly or prove the equivalent assertion that *any finite subset of X is closed.* This is exactly what it means to closed in the finite complement topology. Therefore, X is T_1. However, limits aren't unique (see the example with $X = \mathbb{R}$ and x_n converging to every point in the space).

Recall from univariate analysis, the following argument for proving that the limit of a real sequence is unique: Suppose that there exist $x \neq y$ s.t. $x_n \to x$ and $x_n \to y$ as $n \to \infty$ i.e.

$$\forall \varepsilon > 0,\ \exists N_1 > 0 \text{ such that } n \geqslant N_1 \implies |x_n - x| \leqslant \varepsilon$$
$$\forall \varepsilon > 0,\ \exists N_2 > 0 \text{ such that } n \geqslant N_2 \implies |x_n - y| \leqslant \varepsilon.$$

Let $\varepsilon = |x - y|/3$. Taking $n \geqslant \max\{N_1, N_2\}$, $x_n \in (x - \varepsilon, x + \varepsilon) \cap (y - \varepsilon, y + \varepsilon) = \emptyset$ which is a contradiction.

Proof. (Hausdorff $\implies$ uniqueness of limits). Assume for a contradiction that $x_n \to x$ and $x_n \to y$ with $x \neq y$. As X is T_2, there exist disjoint neighbourhoods U and V of x and y respectively. By the definition of convergence

$$x_n \to x \text{ means that } \exists N_U > 0 \text{ such that } n \geqslant N_U \implies x_n \in U$$
$$x_n \to y \text{ means that } \exists N_V > 0 \text{ such that } n \geqslant N_V \implies x_n \in V.$$

Letting $n \geqslant \max\{N_U, N_V\} \implies x_n \in U \cap V = \emptyset$ which is a contradiction. □

e.g. $X = \mathbb{R}$ with the countable complement topology $\mathcal{T}_c$. This isn't a Hausdorff space because for any distinct points $x, y \in \mathbb{R}$, any open neighbourhoods U, V about x, y would be uncountable in $\mathbb{R}$ (as their complements are countable). Thus, U and V must intersect. However, $(\mathbb{R}, \mathcal{T}_c)$ is T_1 because $\mathcal{T}_f \subseteq \mathcal{T}_c$.

Proposition 1.32. For an infinite set X equipped with the countable complement topology, a sequence converges if and only if it's eventually constant (or stabilises).

Proof. Assume that $x_n \to x$ but x_n doesn't stabilise at x. Take $U = (X \setminus \{x_n\}) \cup \{x\}$ as an open neighbourhood[1] of x. Since x_n doesn't stabilise at x, this means that $x_n \notin U$ for infinitely many n, contradicting $x_n \to x$. The reverse implication is an exercise. □

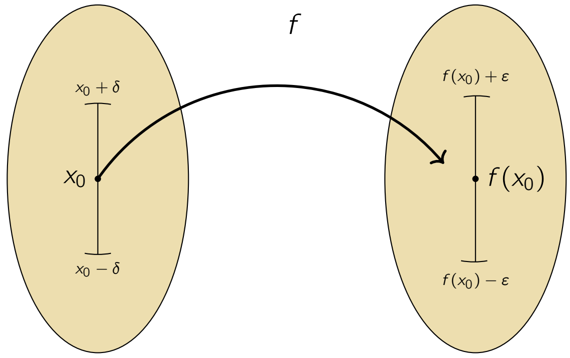
e.g. Let $A = (a, b) \subseteq \mathbb{R} = X$ equipped with the countable[2] complement topology. The claim is that $A' = \overline{A} = \mathbb{R}$. If we let $x \in \mathbb{R}$, an open neighbourhood of x has finite complement (i.e. is an uncountable subset of $\mathbb{R}$) so it must hit any uncountable set. In particular, it hits A. Therefore, $A' = \mathbb{R}$. However, if $x \notin A$ then there does not exist a sequence $\{x_n\} \subseteq A$ with $x_n \to x$ (in $\mathcal{T}_c$). For if it did, it'd have to stabilise at x.

[1] We adjoin the x in case it occurs at some point in the sequence x_n that we've removed.

[2] $\mathcal{T}_c$ does not feel sequences - It happens to be the case that $x \in A' \iff x$ is a limit of a net of elements of A. Nets are generalisations of sequences.

1.6 Continuous Functions

As usual, we'll motivate our definition by considering $\mathbb{R}$. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. Recall that we say f is continuous at x_0 if[3] for every $\varepsilon > 0$, there exists a $\delta_{x_0,\varepsilon} =: \delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| \leqslant \varepsilon$.



From the definition, we see that $f((x_0 - \delta, x_0 + \delta)) \subseteq (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. Further, we can note that if we label $y_0 = f(x_0)$, then $f^{-1}(y_0 - \varepsilon, y_0 + \varepsilon) \supseteq (x_0 - \delta, x_0 + \delta)$, the latter of which is an open set.

Definition 1.33. Let $f\colon X \to Y$ be a map between two topological spaces X, Y. We say f is **continuous** if for every open $U \subseteq Y$, $f^{-1}(U) \subseteq X$ is open in X.

Definition 1.34. We say that $f\colon X \to Y$ is an **open map** if for every open $U \subseteq X$, the image $f(U) \subseteq Y$ is also open in Y.

Remark Both X and Y could be equal as sets but be equipped with different topologies.

1) There's no notion of continuity at a point $x \in X$ in the setting of topological spaces and maps between them.

2) To check continuity of a map, it's sufficient to check the set of basic open neighbourhoods $\mathcal{B}$ that generate the topology on Y i.e.

$$\forall B \in \mathcal{B},\ f^{-1}(B) \text{ is open in } X \qquad (\star)$$

By lemma 1, if $(\star)$ is satsified for every $B \in \mathcal{B}$, we can write any open U in Y as

$$U = \bigcup_{i \in I} B_i \implies f^{-1}(U) = \bigcup_{i \in I} \underbrace{f^{-1}(B_i)}_{\substack{\text{need not}\\ \text{be basic}}} \text{ - which is open in } X.$$

e.g. Let $f\colon \mathbb{R}^k \to \mathbb{R}^m$. We say that f is continuous at $x_0 \in \mathbb{R}^k$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|x_0 - x\| < \delta \implies \|f(x_0) - f(x)\| < \varepsilon$. Then f is continuous if it's continuous at every $x_0 \in \mathbb{R}^k$.

[3] In words, the difference in the abscissa being less than $\delta \implies$ the difference in the ordinate is $< \varepsilon$.

e.g. Consider $f\colon \mathbb{R} \to \mathbb{R}_\ell$, $x \mapsto x$ (where $\mathbb{R}_\ell$ denotes $\mathbb{R}$ equipped with the lower limit topology generated by $\mathcal{B}_\ell = \{[a, b)\colon -\infty < a < b < +\infty\}$). Using remark 2, we can check the continuity by looking at basic elements. The pre-image $f^{-1}([a, b)) = [a, b)$ which isn't open in $\mathbb{R}$ so f isn't continuous. However, f is an open map as for any $U \subseteq \mathbb{R}$ open (in the standard topology), $f(U) = U$ which is open in the lower limit topology as $\mathcal{T}_\ell$ is finer than $\mathcal{T}$.

For a map that's continuous but not open, consider $f\colon \mathbb{R}_\ell \to \mathbb{R}$, $x \mapsto x$.

e.g. Consider $f\colon \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto |x|$. The map f is certainly continuous in the usual sense. However, for $a \neq 0$, $f((-a, a)) = [0, a)$ which isn't open in the standard topology on $\mathbb{R}$ so f isn't open.

e.g. Let X be a topological space and $Y \subseteq X$ a subspace. The inclusion map (or canonical embedding) is $\iota\colon Y \hookrightarrow X$ defined by $y \mapsto y$. Take $U \subseteq X$ open. Note that $\iota^{-1}(U) = Y \cap U$ and this is precisely the definition of an open set in Y equipped with the subspace topology. Indeed, the subspace topology is the coarsest topology on Y that makes the inclusion map continuous.

Theorem 1.35 Let $f\colon X \to Y$ be a map between topological spaces. Then TFAE:

1) f is continuous,
2) For every subset A of X, $f(\overline{A}) \subseteq \overline{f(A)}$ and,
3) For every closed set B in Y, the set $f^{-1}(B)$ is closed in X.
4) For every $x \in X$ and every neighbourhood U of $f(x)$, $\exists V$ neighbourhood of x. Then $f(V) \subseteq U$.

Proof.

$1 \implies 2$ Assume that f is continuous and let $A \subseteq X$. Let $x \in \overline{A}$. We want to show that $f(x) \in \overline{f(A)}$. Let V be a neighbourhood of $f(x)$. Then $f^{-1}(V)$ is an open subset of X containing x and so it must intersect A at some point y. Then V intersects $f(A)$ at $f(y)$ meaning that $f(x) \in \overline{f(A)}$.

$2 \implies 3$ Let B be closed in Y and $A = f^{-1}(B)$. Then $f(A) \subseteq B$. If x is a point of $\overline{A}$,

$$f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B$$

so that $x \in f^{-1}(B) = A$. Thus $\overline{A} \subseteq A$.

$3 \implies 1$ Let B be a closed set in Y. Thus, its complement $Y \setminus B$ is open in Y. By supposition, $f^{-1}(B)$ is closed in X so

$$f^{-1}(Y \setminus B) = f^{-1}(Y) \setminus f^{-1}(B) = X \setminus f^{-1}(B)$$

is an open set in X so f is continuous.

$4 \implies 1$ Let $x \in f^{-1}(U)$ where U is an open neighbourhood of $f(x)$. We want $f^{-1}(U)$ to be open. By 4), there exists an open neighbourhood V_x of x and $f(V_x) \subseteq U$. This last condition tells us that $V_x \subseteq f^{-1}(f(V_x)) \subseteq f^{-1}(U)$ so we can write

$$f^{-1}(U) = \underbrace{\bigcup_{x \in f^{-1}(U)} V_x}_{\text{open}} \implies f \text{ is continuous.}$$

$1 \implies 4$ Suppose that f is continuous. Take $x \in X$ and U as any open neighbourhood of $f(x)$. Then $f^{-1}(U)$ is an open neighbourhood of x. Finally, note that $f(f^{-1}(U)) \subseteq U$.

□

Definition 1.36. Let X and Y be topological spaces and $f\colon X \to Y$ be a bijection. If both f and its inverse are continuous, then f is called a **homeomorphism**. If there exists a homeomorphism between two spaces X and Y, we say that they are homeomorphic and write $X \cong Y$.

Remark In terms of open sets, X and Y being homeomorphic tells us that U is open in X iff its image is open in Y. Thus, a homeomorphism gives a bijective correspondence between the open sets of X and Y. If a property of X is expressed solely in terms of the topology on X, the same property holds for any space homeomorphic to X. Such a property is called a **topological invariant**. Examples include connectedness, cardinality, compactness, metrizability etc. Finally, note that the property of being homeomorphic is an equivalence relation.

Definition 1.37. Suppose that we have an injective continuous map $f\colon X \to Y$. Let $Z = \text{im}(f)$. If we restrict f to a new map $\widetilde{f}\colon X \to Z$ and $\widetilde{f}$ happens to be a homeomorphism, we say that the original map $f\colon X \to Y$ is a **topological embedding** of X in Y.

e.g. The identity map on X, id_X, is a homeomorphism.

e.g. The inclusion map $\iota\colon Y \hookrightarrow X$ is a topological embedding.

e.g. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $x \mapsto 3x + 1$. The map f is clearly continuous and bijective. To show that it's a homeomorphism, consider $g\colon \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto (x-1)/3$. This is clearly continuous and is the inverse of f so f is a homeomorphism.

e.g. Consider the continuous map $f\colon (-1,1) \to \mathbb{R}$, $x \mapsto x/(1-x^2)$. The map $g\colon \mathbb{R} \to (-1,1)$ defined by $g(x) = 2x/(1+\sqrt{1+4x^2})$ is also continuous. Noting that $g \circ f = \text{id}_{(-1,1)}$ and $f \circ g = \text{id}_{\mathbb{R}}$, we conclude that f is a homeomorphism. Therefore, $\mathbb{R} \cong (-1,1)$.

Proposition 1.38. $\mathbb{R} \ncong [-1,1]$

Proof. Let f be a continuous map from $[-1,1]$ to $\mathbb{R}$. By the extreme value theorem, f attains its extrema so f cannot be surjective (and so can't be a bijection so definitely isn't a homeomorphism). □

e.g. Consider $f\colon [0,1) \to S^1$ (where $[0,1)$ and S^1 are endowed with the subspace topology from $\mathbb{R}$ and $\mathbb{R}^2$ respectively) defined by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$.

It's certainly a bijective map and is continuous as its components are continuous. However, f is not an open map. Consider $f([0,1/4))$ which is an arc of the unit circle (including $f(0)$ and excluding $f(1/4)$). There isn't any open neighbourhood about $f(0)$ contained in the image arc so $f([0,1/4))$ isn't open and thus f isn't a homeomorphism.

1.6.1 Locality of Continuity

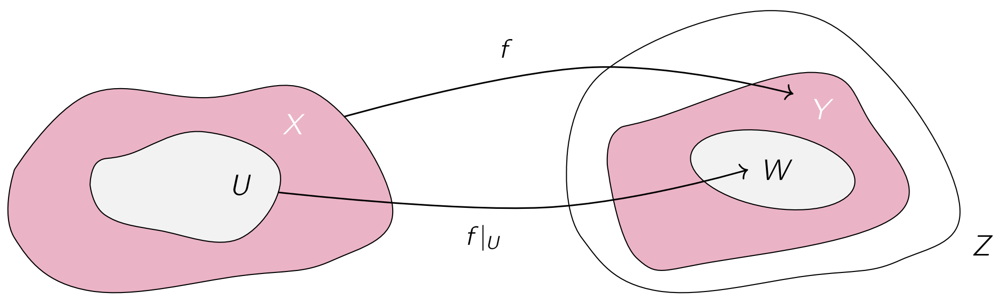
In this section, we discuss the extent to which continuity is a local property (i.e. can we infer global continuity from local continuity). We start with a useful lemma.

Lemma 5 Let $f\colon X \to Y$ be a continuous map between topological spaces. Then

1) If U is a subspace of X, then $f|_U\colon U \to Y$ is continuous.

2) If Z contains Y (i.e. is an ambient space of Y), then $f\colon X \to Z$ is continuous.

3) If W is a subspace of Y that contains $f(X)$, then $f_{\text{restr.}}\colon X \to W$ is continuous.



Proof. 1) Let $V \subseteq Y$ be open. Then $(f|_U)^{-1}(V) = f^{-1}(V) \cap U$ which is open in the subspace topology as $f^{-1}(V)$ is open in X (by the continuity of f). 2) Note that the extended map is a composition of the original $f\colon X \to Y$ and the canonical inclusion $\iota\colon Y \to Z$. This composition is continuous. 3) Let B be open in W i.e. $B = U \cap W$ for some U open in Y. Since $f(X) \subseteq W$, $f^{-1}(U) = (f_{\text{restr.}})^{-1}(U)$. Since $f^{-1}(U)$ is open, $f_{\text{restr.}}$ is continuous. □

Definition 1.39. We call a collection of open subsets $\{U_i\}_{i\in I}$ of X an **open cover of** X if $X = \bigcup_{i\in I} U_i$.

Lemma (**Locality of continuity**) Suppose $f\colon X \to Y$ is a map and that $\{U_i\}_{i\in I}$ is an open cover of X. Then f is continuous $\iff$ $\forall i \in I$, $f|_{U_i}\colon U_i \to Y$ is continuous.

Proof. By (1) of lemma 5, for each $i \in I$, $f|_{U_i}$ is continuous. For the converse, assume that $f|_{U_i}$ is continuous for each $i \in I$. We need to somehow glue the $f|_{U_i}$ to make a continuous function. Let $V \subseteq Y$ be open. Then

$$f^{-1}(V) = \bigcup_{i\in I} f^{-1}(V) \cap U_i = \bigcup_{i\in I} (f|_{U_i})^{-1}(V).$$

As each $(f|_{U_i})^{-1}(V)$ is open in U_i and U_i is open in X, each $(f|_{U_i})^{-1}(V)$ is also open in X. Then an arbitrary union of these open sets is still open in X so f is continuous. □

Lemma (**The Pasting Lemma**) Let $X = A \cup B$ with A, B **closed** in X and $f\colon A \to Y$, $g\colon B \to Y$ be continuous maps that agree on the intersection $A \cap B$. Then the map $h\colon X \to Y$ defined below is continuous.

$$h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B. \end{cases}$$

Proof. Recall that h is continuous iff the pre-image of any closed set C in Y is closed in X. Now note that

$$\begin{aligned} h^{-1}(C) &= \left(h^{-1}(C) \cap A\right) \cup \left(h^{-1}(C) \cap B\right) \\ &= \underbrace{f^{-1}(C)}_{\text{closed in } A} \cup \underbrace{g^{-1}(C).}_{\text{closed in } B} \end{aligned}$$

Being closed in A and A being closed in X means that the original set is closed in X. The same applies when you replace A with B so $h^{-1}(C)$ is closed in X. □

e.g. Let $f\colon [1, \infty) \to \mathbb{R}$ be defined by $x \mapsto x$ and $g\colon (-\infty, 1] \to \mathbb{R}$ be defined by $x \mapsto -x + 2$. Both A and B are closed, $A \cap B = \{1\}$ and $f(1) = g(1)$ so the conditions for the pasting lemma are satisfied. Therefore, $h(x) = |x - 1| + 1$ is a continuous map. (It's also continuous in the usual sense)

e.g. Consider $h\colon \mathbb{Q} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} 0, & x < \sqrt{2} \\ 1, & x > \sqrt{2}. \end{cases}$$

Let $Y = \mathbb{R}$, $A = (-\infty, \sqrt{2}] \cap \mathbb{Q}$ and $B = [\sqrt{2}, \infty) \cap \mathbb{Q}$. Apply the pasting lemma for $f\colon A \to Y$ taking $x \mapsto 0$ and $g\colon B \to Y$ taking x to 1 and you get that h is a continuous map.

1.7 Product Topology ($i \in I$)

We aim to generalise the notion of the product topology that we defined for two sets X, Y to a possibly arbitrary collection $\{X_i\}_{i \in I}$ of topological spaces. So the main question is how we endow a topology onto such a Cartesian product.

In the case of two spaces X_1 and X_2, we declared that the product topology on $X_1 \times X_2$ was generated by the collection $\mathcal{B} = \{U_1 \times U_2 : U_i \text{ is open in } X_i \text{ for } i = 1, 2\}$. A natural suggestion for a topology on X is to simply write out the same definition for the generating collection but make everything indexed by $i \in I$. We'll see that such a topology doesn't satisfy some desirable properties (e.g. if all the component spaces are compact, the box topology on their Cartesian product isn't necessarily compact) as it is simply too fine.

Definition 1.40. The box topology on X is the topology generated by the collection

$$\mathcal{B}_{\text{box}} = \left\{ \prod_{i \in I} U_i : \forall i \in I.\ U_i \text{ is open in } X_i \right\}.$$

Recall that we defined projection maps earlier on and noted that the product topology is the coarsest topology that allows projections to be continuous maps. In the general case, we can define

$$\pi_i : \prod_{j \in I} X_j \to X_i, \qquad (x_j)_{j \in I} \mapsto x_i.$$

Suppose that for some $i_0 \in I$, π_{i_0} is continuous and let U_{i_0} be open in X_{i_0}. Then $\pi_{i_0}^{-1}(U_{i_0})$ has to be open in X. Note that

$$\pi_{i_0}^{-1}(U_{i_0}) = U_{i_0} \times \prod_{\substack{i \neq i_0 \\ i \in I}} X_i.$$

If all projections are to be continuous, then all sets of the form $\pi_i^{-1}(U_i)$ must be open, where i traverses through the whole of I and U_i traverses through all the open sets of X_i. The topology generated by these sets on X is therefore the coarsest for which all projections are continuous.

Ideally, we'd like to define a topological basis with such sets. Accordingly, if we take the intersection of finitely many of these pre-images of projections, we obtain a set of the form

$$U_{i_0} \times \cdots \times U_{i_k} \times \prod_{\substack{i \notin \{i_0, \dots, i_k\} \\ i \in I}} X_i.$$

Therefore, the collection of such sets is closed under finite intersections.

Definition 1.41. The product topology on X is the topology generated by the collection

$$\mathcal{B}_{\text{prod}} = \left\{ \prod_{i \in I} U_i : \forall i \in I.\ U_i \text{ is open in } X_i,\ U_i = X_i \text{ for all but finitely many } i \right\}.$$

Remark This is more restrictive than the box topology in general. However, when I is a finite set both generating sets (and hence topologies) coincide. Also, $\mathcal{B}_{\text{box}}$ is finer than $\mathcal{B}_{\text{prod}}$. Both collections satisfy the intersection property which is stronger than the second axiom for a topological basis. We'll use the product topology by default from now on.

e.g. Consider $X = \mathbb{R}^\omega = \prod_{i=1}^{\infty} \mathbb{R}$. Take $A = \prod_{i=1}^{\infty} (0,1) \subseteq X$.

Notice that A is open in the box topology (and is basic). However, A is not open in the product topology. Suppose that A were open in the product topology on X. Then there exists at least one basic open neighbourhood $U \subseteq A$ that can be written as a product $U = \prod_i U_i$ where $U_i = \mathbb{R}$ for $i \geqslant i_0$. Then $\mathbb{R} = \pi_{i_0}(U) \subseteq \pi_{i_0}(A) = (0,1)$ which is a contradiction.

e.g. For all $i \in \mathbb{N}$, let $X_i = \{0,1\}$ be equipped with the discrete topology and consider

$$X = \prod_{i=1}^{\infty} X_i.$$

Every $x \in X$ is of the form $x = (x_i)_{i=1}^{\infty}$ where $x_i \in \{0,1\}$ and we know that each $\{x_i\}$ is open in X_i so $\{x\}$ is open in the box topology. Accordingly, $\mathcal{T}_{\text{box}}$ coincides with $\mathcal{P}(X)$.

Now equip X with the product topology. We wish to describe the product topology. Doing so on X isn't straight-forward so we'll translate the problem from sequences in X to decimal representations in $[0,1]$ via the map $f : X \to [0,1] \subseteq \mathbb{R}$ defined by

$$(b_i)_{i \in \mathbb{N}} \mapsto \sum_{i=1}^{\infty} \frac{b_i}{2^i}.$$

If the map is 'good enough', describing the topology induced by f on $[0,1]$ is enough to describe the topology on X. The map f is "almost bijective". It's certainly surjective and if we remove the dyadic numbers $\mathcal{D} := \{k/2^n\}$ from $[0,1]$ and their pre-images under f from X, the restricted map $f|_{\text{new}}$ is injective. The topology on $[0,1] \setminus \mathcal{D}$ is certainly not discrete and therefore can't be induced by a discrete map.

Claim The topology on $[0,1] \setminus \mathcal{D}$ coincides with the subspace topology induced from $\mathbb{R}$.

Proof. The basic open neighbourhoods of $[0,1] \setminus \mathcal{D}$ contain elements whose decimal representations are prescribed finitely bits and the rest are free. Since we've removed $\mathcal{D}$ from $[0,1]$, these intervals are open. It's a fact that any interval $(a,b) \subseteq [0,1]$ can be written as a disjoint union of dyadic intervals i.e. $(a,b) = \coprod_{i \in I}(a_i, b_i)$ where $a_i, b_i \in \mathcal{D}$. In particular, the claim follows. □

This topology is finer than the standard topology but coarser than the lower limit topology.

e.g. An exercise is to try the prior example with $\{0,1,\ldots,9\}^N$ instead of $\{0,1\}$.

e.g. Let $I = [0,1]$, $X = \prod_{i \in I} \mathbb{R} = \{f : I \to \mathbb{R}\}$ and $A = \{f \in X : f \text{ is continuous}\}$.

Suppose that $f \in \overline{A}$. This means that every neighbourhood of f intersects A non-emptily i.e we can approximate f by elements of A. (Note that the topology on X determines the accuracy to which we can approximate f.)

In the box topology, we claim that $\overline{A} = A$. In other words, no discontinuous functions can be approximated by continuous functions. Let $f \notin A$ (be discontinuous). Then $\exists x_0 \in [0,1]$ such that

$$\underbrace{\liminf_{x \to x_0} f(x)}_{y_1} < \underbrace{\limsup_{x \to x_0} f(x)}_{y_2}.$$

Let x_{n_j} be a sequence in $[0,1]$ that converges to x_0 as $j \to \infty$ and whose image sequence $f(x_{n_j}) \to y_0$ as

$j \to \infty$. Assume that all the x_{n_j} are distinct and define $\varepsilon = |y_2 - y_1|/3$. Define

$$U = \prod_{n=1}^{\infty} (f(x_{n_1}) - \varepsilon, f(x_{n_1}) + \varepsilon) \ \times \ \prod_{n=1}^{\infty} (f(x_{n_2}) - \varepsilon, f(x_{n_2}) + \varepsilon) \ \times \ \prod_{x \notin \{x_{n_1}, x_{n_2}\}} \mathbb{R}.$$

Let $g \in U$. Then we can say that

$$\liminf_{x \to x_0} g(x) \leqslant \liminf_{x \to x_0} f(x) + \varepsilon < \limsup_{x \to x_0} f(x) - \varepsilon \quad \leqslant \limsup_{x \to x_0} g(x).$$

Therefore, $U \cap A = \emptyset \implies f \notin \overline{A}$.

In the product topology, we claim that $\overline{A} = X$. Let $f \in X$ and U be any neighbourhood of f. (Every open set is a union of basis elements so we can assume that U is basic.) Take

$$U = \prod_{x \notin \{x_1, \ldots, x_n\}} \mathbb{R} \times \prod_{i=1}^{n} U_{x_i}.$$

We can find a piecewise linear function g so that for each $i \leqslant n$, $g(x_i) = f(x_i)$. Since $g \in U \cap A$, we can conclude that $f \in \overline{A} \implies \overline{A} = X$.

Theorem 1.42

1) If $X = \prod_{i \in I} X_i$, where for all $i \in I$, $\mathcal{B}_i$ is a general topology on X_i, then

   (a) the box topology on X is given by

$$\left\{ \prod_{i \in I} B_i : \forall i \in I.\ B_i \in \mathcal{B}_i \right\}.$$

   (b) the product topology on X is given by

$$\left\{ \prod_{i \in \{i_1, \ldots, i_n\}} B_i \times \prod_{i \notin \{i_1, \ldots, i_n\}} X_i : \forall j \leqslant n.\ B_{i_j} \in \mathcal{B}_{i_j} \right\}.$$

2) Let $i \in I$ and for each $i \in I$, let $A_i \subseteq X_i$ be a subspace. Define $A := \prod_{i \in I} A_i$. Then the product topology of the subspaces $A_i \subseteq X_i$ coincides with the subspace topology of $A \subseteq X$ (where X is equipped with either the box or product topology).

3) If for each $i \in I$, X_i is Hausdorff, then X is Hausdorff (in the box and product topologies).

4) If for each $i \in I$, $A_i \subseteq X_i$ and A is of the form $\prod_{i \in I} A_i$, we have that

$$\overline{\prod_{i \in I} A_i} = \overline{A} = \prod_{i \in I} \overline{A_i}.$$

Proof. Exercise. □

Remark For lack of better notation, denote the closures of A in the box and product topologies on X by $\overline{A}_{\text{box}}$ and $\overline{A}_{\text{prod}}$ respectively. For every set A of the form $\prod_{i \in I} A_i$, these closures coincide. However, this doesn't hold in general. As for why, recall that if we know the closures of all sets in a space, we can completely describe the topology. If the equality did hold, both the box and product topologies on X would coincide in general but we know that this isn't the case.

Theorem 1.43 Let $f\colon A \to X = \prod_{i \in I} X_i$ be defined by $a \mapsto (f_i(a))_{i \in I}$.

If X is endowed with the product topology, then f is continuous $\iff$ $\forall i \in I$, f_i is continuous. If X is endowed with the box topology instead, only the forward implication holds.

Proof. The forward implication is the same as in the finite case (which is addressed in an assignment that I'm yet to do).

For the reverse implication, suppose that all the f_i are continuous. It's sufficient to show that the pre-image of every basic open neighbourhood $U \subseteq X$ under f is open in A. Note that U is of the form

$$U_{i_1} \times \cdots \times U_{i_n} \times \prod_{i \notin \{i_1, \ldots, i_n\}} X_i.$$

Therefore, we can write that

$$f^{-1}(U) = \left\{a \in A \colon \forall i \in \{i_1, \ldots, i_n\},\ f_i(a) \in U_i\right\} = \bigcap_{i=i_1}^{i_n} \underbrace{f_i^{-1}(U)}_{\vdots}$$

Which is a finite intersection of open sets (due to the continuity of each f_i) and is therefore open itself. $\square$

1.8 Metric Topology

Definition 1.44. Let X be a set. A **metric** on X is a map $d\colon X \times X \to \mathbb{R}$ such that $\forall x, y, z \in X$:

i) $d(x, y) \geqslant 0$ and $d(x, y) = 0 \iff x = y$,

ii) $d(x, y) = d(y, x)$,

iii) $d(x, z) \leqslant d(x, y) + d(y, z)$.

Remark The first condition is called positivity and that $d(x, y) = 0$ is equivalent to $x = y$ is a property known as the separation of points. The reason for such a name is that every distinct pair of points should be separated by a positive distance. The second is called symmetry and the final property is the triangle inequality.

e.g. Let $X = \mathbb{R}^n$ and define for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$,

$$d_2(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2}.$$

Definition 1.45. Given $x \in X$ and $r > 0$, $\mathbb{B}_d(x, r) = \{y \in X \colon d(x, y) < r\}$ is the **open ball** in the metric d centered at x with radius r.

e.g. Let $X = \mathbb{R}^n$ and $p > 0$. Define the p-metric by $d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}$.

It isn't immediately clear that this is indeed a metric. Positivity and symmetry are straight-forward but the triangle inequality is known as a theorem called Minkowski's inequality.

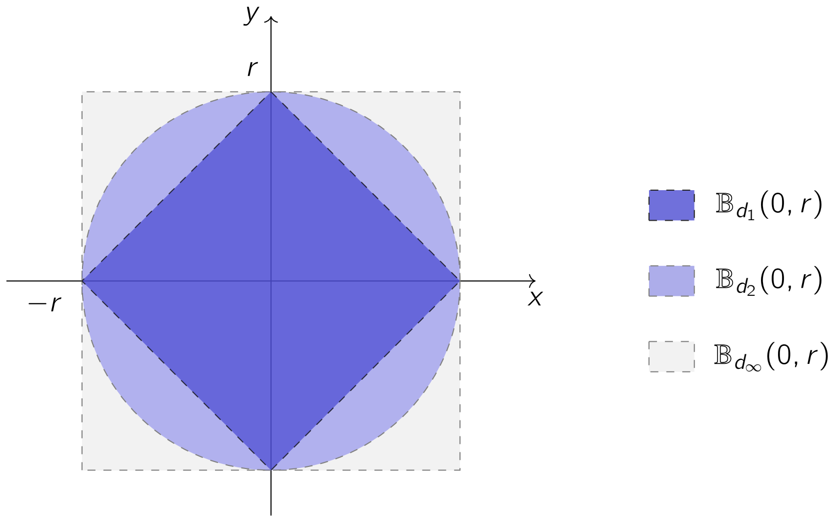


Figure 1.4: An illustration of the metric balls (superimposed) centered at the origin with radius r in $\mathbb{R}^2$ with respect to the metrics d_1, d_2 and d_∞.

e.g.

$$\mathcal{L}^2(\mathbb{R}) = \left\{\text{measurable } f : \mathbb{R} \to \mathbb{R} : \int_{\mathbb{R}} |f|^2 < \infty\right\}$$

$$\forall f, g \in \mathcal{L}^2(\mathbb{R}), \text{ define } d(f, g) = \left(\int_{\mathbb{R}} |f - g|^2\right)^{1/2}$$

This map is certainly non-negative and symmetric. The triangle inequality comes from the triangle inequality on $(\mathbb{R}, |\cdot|)$. However, when it comes to the separation of points (in this case, the functions are the 'points' in $\mathcal{L}^2(\mathbb{R})$), we see that although f and g may differ, $d(f, g)$ could vanish. For instance, if we let f be identically 1 on $\mathbb{R}$ and g be identically 1 on $\mathbb{R} \setminus \{x\}$ where $g(x) = 0$, f and g certainly aren't equal but $d(f, g) = 0$.

It's clear where the problem lies. Functions that differ on a set whose width[4] (e.g. a singleton) doesn't contribute to the integral aren't distinguishable by the integral (metric).

[4]After some careful considerations in building up the theory of size, we refer to this as the measure of a set. Most introductory books on measure theory go into it well and I'm not well-versed enough to give an explanation here.

To rectify this issue, we can define an equivalence relation on $\mathcal{L}^2(\mathbb{R})$ where $f \sim g$ iff f and g differ on at most a set of measure 0. Now we can define a new space of equivalence classes of such functions, denoted $L^2(\mathbb{R}) = \mathcal{L}^2(\mathbb{R})/\sim$.

Equipping this space with the same map d turns this map into a metric, say $\tilde{d}$. The problem with separating functions is now gone. Note that when using this new map $\tilde{d}$, the inputs are equivalence classes, e.g. we compute $\tilde{d}([f],[g])$. However, confusion often does not arise from taking a representative from each equivalence class and we end up with the original definition of d for the purpose of calculations.

e.g. Let X be any set and $d(x,y) = 0$ if $x = y$ and $d(x,y) = 1$ if $x \neq y$. Non-negativity, positivity and symmetry are all immediate from the definition. For the triangle inequality

$$d(x,z) \leqslant d(x,y) + d(y,z),$$

notice that the LHS is always $\leqslant 1$. Therefore, the RHS $\leqslant$ LHS iff RHS $= 0$ which is equivalent to $x = y = z$, thereby implying that $d(x,z) = 0$.

This metric is called the discrete metric on X. The open balls that the discrete metric admits are (for any $x \in X$) $\mathbb{B}_d(x,r) = X$ if $r > 1$ and $\mathbb{B}_d(x,r) = \{x\}$ if $r \leqslant 1$.

Definition 1.46. If X is a set and $d : X \times X \to \mathbb{R}$ is a metric, the metric topology associated to X is the topology generated by $\mathcal{B}_d = \{\mathbb{B}_d(x,r) : x \in X,\ r > 0\}$. (X,d) is called a metric topological space.

The definition assumes that $\mathcal{B}_d$ is a topological basis. Let's prove it.

i) For every $x \in X$ and every $r > 0$, $x \in \mathbb{B}_d(x,r)$ so $\mathcal{B}_d$ is a cover of X.

ii) Let $z \in \mathbb{B}_d(x,r_1) \cap \mathbb{B}_d(y,r_2)$. Although the picture is somewhat misleading from a geometric perspective, it motivates the idea of what's going on!

Take $0 < \varepsilon < \min\{r_1 - d(z,x), r_2 - d(z,y)\}$. Let ω be an arbitrary point in $\mathbb{B}_d(z,\varepsilon)$. We want to show that ω is an element of both $\mathbb{B}_d(x,r_1)$ and $\mathbb{B}_d(y,r_2)$ i.e. that $d(\omega,x) < r_1$ and $d(\omega,y) < r_2$. As a consequence of the triangle inequality,

$$\begin{aligned} d(\omega,x) &\leqslant d(\omega,z) + d(z,x) \\ &< \varepsilon + d(z,x) \\ &< r_1 - d(z,x) + d(z,x) = r_1 \end{aligned}$$

$$\begin{aligned} d(\omega,y) &\leqslant d(\omega,z) + d(z,y) \\ &< \varepsilon + d(z,y) \\ &< r_2 - d(z,y) + d(z,y) = r_2 \end{aligned}$$

Therefore, $\mathbb{B}_d(z,\varepsilon)$ is such an element of $\mathcal{B}_d$ that contains z and is contained in $\mathbb{B}_d(x,r_1) \cap \mathbb{B}_d(y,r_2)$. Thus, $\mathcal{B}_d$ is a topological basis.

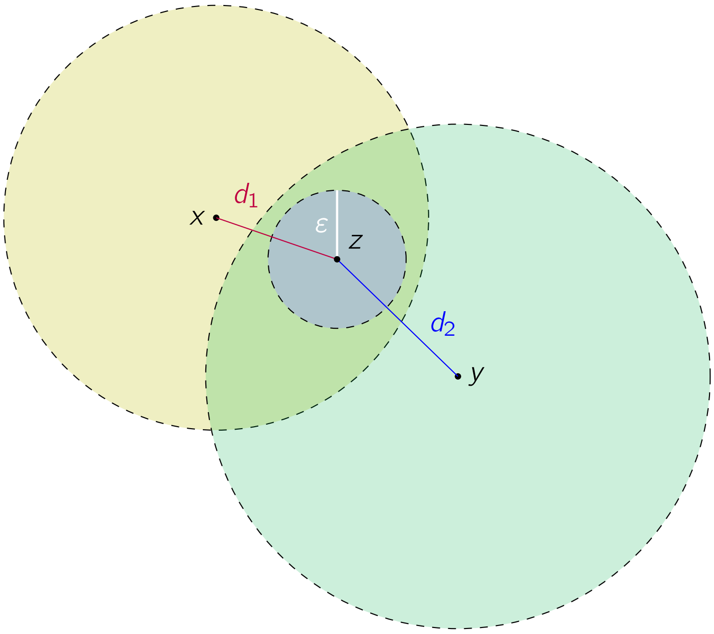


Figure 1.5: In the diagram, $d_1 = d(z, x)$ and $d_2 = d(z, y)$. We want to find an ε such that $\mathbb{B}_d(z, \varepsilon)$ is contained in the intersection.

Explicitly, $U \subseteq X$ is open if $\forall x \in U$, $\exists r > 0$ so that $\mathbb{B}_d(x, r) \subseteq U$. Our regular notion of a space being open in $\mathbb{R}^n$ is a special case of this definition with the metric balls being standard balls. Our definition needs not for the metric balls to be centered at x a priori[5]. We could equally well take a metric ball containing x but centered elsewhere. Then, we can fit a smaller metric ball centered at x inside the prior ball.

Proposition 1.47. Suppose that (X, d) is a metric topological space and that $\{x_n\} \subseteq X$ is a sequence in X. Then $x_n \to x$ as $n \to \infty$ iff $d(x_n, x) \to 0$ as $n \to \infty$ (as a sequence of real numbers).

Proof. Exercise. □

Thus, in the case of a metric topological space, we can translate questions of convergence to the more familiar setting of $\mathbb{R}$.

e.g. Let $X = \mathbb{R}^n$ be equipped with d_2. A set is open if and only if we can squeeze a metric ball about every point it contains. The metric balls that d_2 admits are the standard balls we're used to seeing in $\mathbb{R}^n$. Therefore, d_2 generates the standard topology on $\mathbb{R}^n$.

e.g. Consider $L^2(\mathbb{R})$ equipped with the metric from before. Then we can say that as $n \to \infty$, $f_n \to f \iff d(f_n, f) \to 0$.

e.g. Let X be an arbitrary set and d be the discrete metric. The open metric balls as before are X and $\{x\}$. Since every singleton is open, the discrete metric generates the discrete topology on X.

[5] In a way based on theoretical deduction rather than empirical observation.

Given two metrics on a space, how can we compare the topologies they generate?

Lemma 3 (Metric Version) Suppose that d, d' are two metrics on a set X generating the topologies $\mathcal{T}_d, \mathcal{T}_{d'}$ respectively. Then TFAE:

1) $\mathcal{T}_{d'}$ is finer than $\mathcal{T}_d$
2) $\forall x \in X\ \forall \varepsilon > 0,\ \exists \delta > 0$ such that $\mathbb{B}_{d'}(x, \delta) \subseteq \mathbb{B}_d(x, \varepsilon)$.

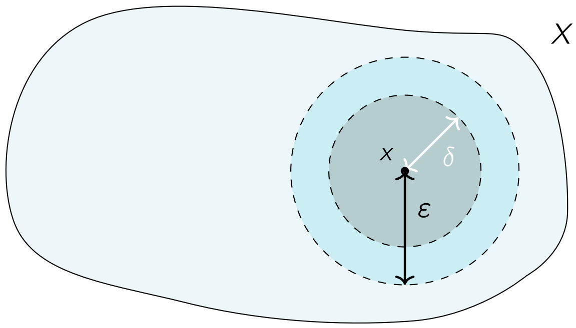


Figure 1.6: An illustration of the metric version of lemma 3.

Remark Note that both metric balls in the lemma are centered at the same point x. From an earlier remark, if we test only balls centered at the same point, the lemma still holds. This is highly useful in narrowing down the metric balls we need to check.

e.g. Let $X = \mathbb{R}^n$ equipped with the d_∞ metric[6] defined by $d_\infty(x, y) = \max_{1 \leqslant i \leqslant n}\{|x_i - y_i|\}$. Which topology does this metric generate? To see which metric it generates, we can use the metric version of lemma 3. Note that

$$d_\infty(x, y) \leqslant d_2(x, y) \leqslant \sqrt{n}\ d_\infty(x, y).$$

This implies that

$$\mathbb{B}_{d_\infty}(x, r) \subseteq \mathbb{B}_{d_2}(x, r) \subseteq \mathbb{B}_{d_\infty}(x, r\sqrt{n}).$$

By the metric version of lemma 3, d_∞ and d_2 generate the same topology on $\mathbb{R}^n$, the standard topology.

Given a metric, we can talk about the topology generated by the metric, the metric topology. However, given a topology, can we say that there exists a metric which generates it?

Definition 1.48. Let $(X, \mathcal{T})$ be a topological space. We say that $\mathcal{T}$ is **metrizable** if there exists a metric d on X that generates T.

e.g. As we just saw, $\mathbb{R}$ equipped with the standard topology is metrizable. Corresponding metrics include d_2 and d_∞.

[6] Check that it is indeed a metric. It's commonly called the uniform/supremum metric.

1.8.1 Metrizability of $\mathbb{R}^\omega$

How about $\mathbb{R}^\omega$? Is it metrizable with respect to the product and (less importantly, post factum) box topology? We don't have a metric so we need to construct one. Given $x = (x_i)_{i\geqslant 1}, y = (y_i)_{i\geqslant 1} \in \mathbb{R}^\omega$, our map needs to incorporate the coordinates, satisfy the properties of a metric and intuitively output a small (resp. large) value when x and y are close (resp. distant).

For a naïve attempt at a construction, consider both $\mathfrak{c}(x, y) = \sup_{i\in\mathbb{N}}\{|x_i - y_i|\}$ and $\mathfrak{p}(x, y) = \sum_{i=1}^{\infty} |x_i - y_i|$. The problem with both of these is that they might not be finite for every $x, y \in \mathbb{R}^\omega$.

It turns out that with respect to the product topology, $\mathbb{R}^\omega$ is metrizable. However, w.r.t. the box topology, $\mathbb{R}^\omega$ isn't metrizable. We state a more general theorem below:

Theorem 1.49 Let $\{X_i\}_{i\in\mathbb{N}}$ be a countable family of metrizable spaces. Then the product space X is metrizable.

$$X = \prod_{i=1}^{\infty} X_i$$

Remark As a reminder, when we write product space, we implicitly equip the space with the product topology.

Proof. First of all, we construct a metric on X. For each $i \in \mathbb{N}$, X_i is metrizable so let d_i be the metric on X_i that generates X_i's metric topology. For $x_i, y_i \in X_i$, define $\overline{d_i}(x_i, y_i) = \min\{d_i(x_i, y_i), 1\}$. By the homework, $\overline{d_i}$ is a metric on X_i that generates the same topology on X_i as d_i.

For $x = (x_i)_{i\in\mathbb{N}}, y = (y_i)_{i\in\mathbb{N}} \in X$, define

$$\mathfrak{d}(x, y) = \sum_{i=1}^{\infty} \frac{\overline{d_i}(x_i, y_i)}{2^i} < \infty.$$

The map $\mathfrak{d}$ is a metric on X. Properties (i) and (ii) are simple to verify and the triangle inequality is proven by applying it in each coordinate in X_i as $\overline{d_i}$ is a metric.

We could equally have defined the following[7] two metrics[8]

$$\widetilde{\mathfrak{d}}(x, y) = \sup_{i\in\mathbb{N}} \left\{ \frac{\overline{d_i}(x_i, y_i)}{i} \right\} \qquad \mathfrak{d}^\star(x, y) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}\, 2^{-i}.$$

[7] For the second metric, the triangle inequality can be proven by noting that $x/(1 + x)$ is convex.
[8] Note that in $\mathfrak{d}^\star$, the d_i aren't necessarily bounded so we consider the quotient $d_i/(1 + d_i)$.

Claim The metrics $\mathfrak{d}$, $\widetilde{\mathfrak{d}}$ and $\mathfrak{d}^\star$ generate the product topology on X.

Denote the topology generated by $\mathfrak{d}$ by $\mathcal{T}'$. Since we can't suppose that the product topology $\mathcal{T}$ on X is metrizable (as that's what we're trying to prove), we'll need the original version of lemma 3 to compare their bases. Thus, we need to show that

- for any basic open neighbourhood U in $\mathcal{T}$ of $x^0 = \{x_i^0\}_{i\in\mathbb{N}}$, there exists an $r > 0$ s.t. $\mathbb{B}_\mathfrak{d}(x^0, r) \subseteq U$, and
- for any $x^0 \in X$ and $r > 0$, there exists a basic open neighbourhood U in the product topology such that $x^0 \in U \subseteq \mathbb{B}_\mathfrak{d}(x^0, r)$.

Let U be a basic open neighbourhood of x^0 in the product topology. Then U is of the form

$$U = \prod_{i=1}^{\infty} U_i$$

where for each $i \in \mathbb{N}$, U_i is a neighbourhood of x_i^0 and $U_i = X_i$ for all $i \notin \{i_1, \ldots, i_k\}$. Since each X_i is metrizable, for each $n \in \{1, \ldots, k\}$, let $\varepsilon_n > 0$ be such that

$$\mathbb{B}_{\overline{d_{i_n}}}(x_{i_n}^0, \varepsilon_n) \subseteq U_{i_n}.$$

Define

$$\varepsilon < \underbrace{\min_{1\leqslant n\leqslant k}\left\{\frac{\varepsilon_n}{2^{i_n}}\right\}}_{\vdots}.$$

Here, we use that the product topology is equipped to X. If we didn't have the product topology equipped, there'd be no constraint on the number of elements we're taking a minimum of in this set so we could end up with $\varepsilon < 0$.

Subclaim $\mathbb{B}_\mathfrak{d}(x^0, \varepsilon) \subseteq U$

Let $y \in \mathbb{B}_\mathfrak{d}(x^0, \varepsilon)$. Then for any $n \in \{1, \ldots, k\}$,

$$\frac{\overline{d_i}(x_{i_n}^0, y_{i_n})}{2^{i_n}} \leqslant \sum_{i=1}^{\infty} \frac{\overline{d_i}(x_i^0, y_i)}{2^i} = \mathfrak{d}(x^0, y) < \varepsilon < \frac{\varepsilon_n}{2^{i_n}}.$$

i.e. for any $n \in \{1, \ldots, k\}$, $y \in \mathbb{B}_{\overline{d_{i_n}}}(x_{i_n}^0, \varepsilon_n) \subseteq U_{i_n}$ and we conclude that $y \in U$.

Conversely, we are to show that every metric ball $\mathbb{B}_\mathfrak{d}(x^0, r)$ contains a neighbourhood (in the product topology) of x^0. We can assume WLOG[9] that $r < 1$ and define $K = \lfloor \log(r/4) \rfloor + 1$. We construct a neighbourhood U of x^0

$$U = \prod_{i=1}^{K} \mathbb{B}_{\overline{d_i}}(x_i^0, r/2) \times \prod_{i=K+1}^{\infty} X_i.$$

[9]Without loss of generality!

Let $y \in U$. Then

$$\begin{aligned}\mathfrak{d}(x^0, y) &= \sum_{i=1}^{K} \frac{\overline{d_i}(x_i^0, y_i)}{2^i} + \sum_{i=K+1}^{\infty} \frac{\overline{d_i}(x_i^0, y_i)}{2^i} \\ &\leqslant \sum_{i=1}^{K} \frac{r}{2^{i+1}} + \sum_{i=K+1}^{\infty} \frac{1}{2^i} \\ &< \frac{r}{2} + \frac{r}{2^{-\lfloor \log(r/4) \rfloor + 1}} \\ &\leqslant \frac{r}{2} + \frac{r}{2} = r\end{aligned}$$

so $U \subseteq \mathbb{B}_{\mathfrak{d}}(x^0, r)$. Both topologies are therefore equal by lemma 3 and so X equipped with the product topology is metrizable, thereby completing the proof. □

1.9 Quotient Topology

The idea of what it means to take the quotient of space in a geometric sense is that one 'glues together' points in order to form another object. Abstractly, this notion of 'gluing' involves identifying points with each other by declaring them to be equivalent.

For example, if we take the unit interval $[0, 1]$ and declare the end-points to be equivalent, the resulting space $[0, 1]/\sim$ (which we call the quotient space of $[0, 1]$ under the equivalence relation $\sim$) is homeomorphic to the unit circle, S^1. A simple case of a slightly more sophisticated construction (called the wedge sum[10]) involves gluing two circles at a single basepoint to obtain a new space (denoted $S^1 \wedge S^1$). More examples of such constructions follow after a few key definitions!

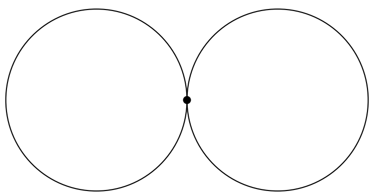


Figure 1.7: The wedge sum of two circles, $\bigwedge_{i=1}^{2} S^1$.

Let X be a topological space, $\sim$ an equivalence relation on X and denote $Y = X/\sim$. How do we endow a topology on Y that's induced by X?

Definition 1.50. A map $f : X \to Y$ is called a quotient map[11] if it is surjective and a subset U of Y is open if and only $f^{-1}(U)$ is open in X.

[10] It's also referred to as a one point union.

[11] Equivalently, f is a quotient map if it is onto and Y is equipped with the quotient topology.

Definition 1.51. Let X be a topological space and $\sim$ an equivalence relation on X. The natural **quotient map** is

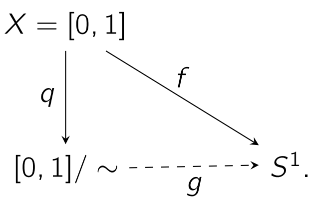
$$\begin{aligned} q\colon X &\longrightarrow X/\sim \\ x &\longmapsto [x]. \end{aligned}$$

We define a topology on $X/\sim$ by declaring that $U \subseteq X/\sim$ is open iff $q^{-1}(U) \subseteq X$ is open. We call this topology the **quotient topology**. Note that the quotient topology automatically makes the quotient map q continuous (and open).

Remark The quotient topology[12] is the **finest** topology such that q is continuous. Any other topology that lets q be continuous is coarser than the quotient topology by definition.

e.g. The following are examples of quotient spaces (and their maps).

1) Let $f\colon X = [0,1] \to S^1$ be defined by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$. Define the equivalence relation $\sim$ on X by $0 \sim 1$ and otherwise no two elements of X are equivalent. Consider the diagram:



The map f is certainly continuous, onto and injective apart from $f(0) = f(1)$. Also consider the quotient map $q\colon [0,1] \to [0,1]/(0 \sim 1)$. Thus, f induces a bijection g such that $f = g \circ q$. One can prove that g is continuous iff $g \circ q$ is continuous[13]. Since $f = g \circ q$, then g is continuous.

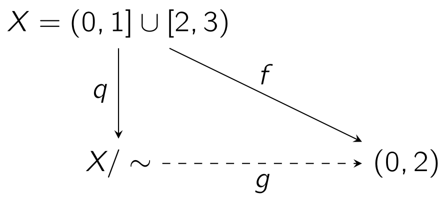
Compact-to-Hausdorff Lemma Suppose that X is compact, Y is Hausdorff and $f\colon X \to Y$ is a continuous bijection. Then f is a homeomorphism.

Proof. This theorem wasn't actually stated in the lectures and I won't prove it here. However, a proof can be found on page 135 under Proposition 13.26 in [2]. □

We'll also need to observe that any subspace Y of a Hausdorff space X is also Hausdorff. Simply note that for any two $x, y \in Y$, they are also elements of X so there exist disjoint open neighbourhoods U, V in X of x and y respectively. Taking $Y \cap U$ and $Y \cap V$ which are disjoint and open (in the subspace topology) neighbourhoods of x and y respectively, we conclude that Y is also Hausdorff. Since $S^1 \subseteq \mathbb{R}^2$, $\mathbb{R}^2$ is Hausdorff and $[0,1]/\sim$ is compact (as the continuous image of a compact space, as we shall see later), it follows from the prior lemma that g is a homeomorphism.

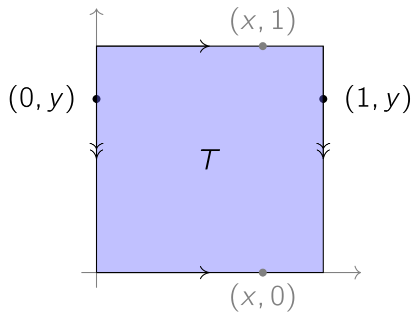
[12] Sometimes also called the final topology of the family of maps $\{q\}$.
[13] The proof of this is straight-forward!

2) Let $X = (0,1] \cup [2,3)$ and $Y = X/(1 \sim 2)$.

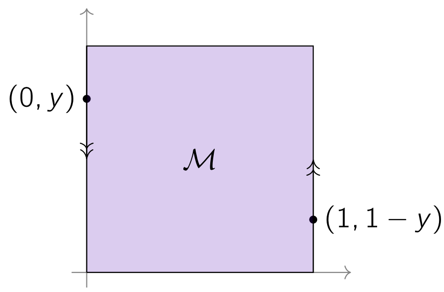


We can define $q(x) = x$ when $x \in (0,1]$ and $q(x) = x - 1$ when $x \in [2,3)$ so that $q(1) = q(2)$ and subsequently obtain a homeomorphism g.

3) The torus T can be represented by a quotient square, i.e. $T = [0,1]^2/\sim$ where $\sim$ is defined by $(0,y) \sim (1,y)$ and $(x,0) \sim (x,1)$ for all $x, y \in [0,1]$.



4) Similarly for the Möbius strip, we can identify sides of a quotient square so that $\mathcal{M} = [0,1]^2/\sim$ where $(0,y) \sim (1,1-y)$ for all $y \in [0,1]$.



Chapter 2

Connectedness and Compactness

2.1 Connected Spaces

Definition 2.1. Let X be a topological space. We say that a **separation** of X is two clopen and disjoint sets A, B whose union $A \cup B = X$. A set X is **disconnected** if there exists a separation of X. A separation is **trivial** if either $A = \emptyset$ or $B = \emptyset$. We say that X is **connected** if every separation of X is trivial.

Remark Note that X being connected is equivalent to X not having any non-trivial clopen subsets.

e.g.

1) Consider $(X, \mathcal{P}(X))$. Suppose that $|X| > 1$. Then $\{x\}$ and its complement are clopen, disjoint and their union is the whole of X i.e. there exists a separation so X isn't connected. X is connected iff $|X| \leqslant 1$.

2) X equipped with the trivial topology $\mathcal{T} = \{\emptyset, X\}$ has no non-trivial clopen subsets so is connected.

3) $X = \mathbb{R}$ equipped with the standard topology is connected as it has no non-trivial clopen subsets.

4) Consider $X = \mathbb{R}_\ell$ and take an open set $[\alpha, \beta)$. The complement $X \setminus [\alpha, \beta) = (-\infty, \alpha) \cup [\beta, \infty)$ is closed (by definition) and open because $\mathcal{T}_\ell$ is finer than the standard topology on $\mathbb{R}$ so $(-\infty, \alpha) \in \mathcal{T}_\ell$. Thus, $\mathbb{R}_\ell$ is not connected.

5) $X = [1, 2] \cup [3, 4]$ is not connected. Take $A = [1, 2]$ and $B = [3, 4]$.

6) Consider $\mathbb{Q} \subseteq \mathbb{R}$.

   Claim $\mathbb{Q}$ (with the subspace topology induced from $\mathbb{R}$) is not connected.

   Let $\alpha < \beta$ both be irrational. Then $A = (\alpha, \beta) \cap \mathbb{Q} = [\alpha, \beta] \cap \mathbb{Q}$ is clopen so $\mathbb{Q}$ is not connected. In fact, $\mathbb{Q}$ is **totally disconnected** i.e. it has no connected subspaces other than singletons.

7) Consider $X = \{a, b\}$ equipped with any topology that isn't discrete.

Claim X is connected.

Proof. Suppose that X isn't connected i.e. there exists a separation A, B. WLOG, let $A = \{a\}$ be clopen. Then $\{b\} = X \setminus \{a\}$ is open so the topology on X is discrete. □

Lemma Let X be a topological space and $Y \subseteq X$ be a subspace. Then a pair of disjoint sets A, B whose union is Y is a separation of Y if and only if A nor B contain the limit points of the other (in X).

e.g.

1) Let $Y = (0, 1) \cup (1, 2) \subseteq \mathbb{R}$. Denote $A = (0, 1)$ and $B = (1, 2)$. Then we see that $A' = \overline{A} = [0, 1]$ and $B' = \overline{B} = [1, 2]$ so that $A' \cap B = B' \cap A = \emptyset$. Therefore, A, B is a separation of Y.

2) Consider $Y = (0, 1) \cup [1, 2) \subseteq \mathbb{R}$. Denote $A = (0, 1)$ and $B = [1, 2)$. Since $A' = \overline{A} = [0, 1]$ and $A' \cap B = \{1\} \neq \emptyset$, we conclude that A, B isn't a separation. Note that in this example, we just re-wrote $Y = (0, 2)$ which we know is connected.

Proof. Let $Y = A \cup B$ be a separation. Then

$$\begin{aligned} B \cap A' &= (B \cap Y) \cap A' \\ &= B \cap (Y \cap A') \\ &\subseteq B \cap (Y \cap \overline{A}) \\ &= B \cap \overline{A}_Y \\ &= B \cap A \\ &= \emptyset. \end{aligned}$$

Similarly, $B' \cap A = \emptyset$.

For the converse, let A and B satisfy the postulated properties.

Claim A is closed in Y.

$$\overline{A} \cap B = (A \cup A') \cap B = \underbrace{(A \cap B)}_{\emptyset} \cup \underbrace{(A' \cap B)}_{\emptyset} = \emptyset$$

Therefore,

$$\overline{A}_Y = \overline{A} \cap Y = \overline{A} \cap (A \cup B) = (\overline{A} \cap A) \cup (\overline{A} \cap B) = A \cup \emptyset = A$$

so A is closed in Y. By symmetry, B is also closed in Y so $Y = A \cup B$ is a separation. □

Theorem 2.2 Let $\{X_i\}_{i \in I}$ be a family of connected topological subspaces of a set X. If their intersection is non-empty i.e. $\exists x_0 \in \bigcap_{i \in I} X_i$, then their union $\bigcup_{i \in I} X_i$ is connected.

Proof. Assume (WLOG) that $X = \cup_{i \in I} X_i$ and let A, B be a separation of X. We wish to show that any separation of X is trivial. By symmetry, let $x_0 \in A$.

Claim For each $i \in I$, $X_i \subseteq A$.

Proof. Since A, B is a separation of X, consider

$$X_i = \underbrace{(A \cap X_i)}_{\widetilde{A}} \cup \underbrace{(B \cap X_i)}_{\widetilde{B}}.$$

Since A and B are disjoint, so are $\widetilde{A}$ and $\widetilde{B}$. Both $\widetilde{A}$ and $\widetilde{B}$ are open in X_i (in the subspace topology). Thus, we have a separation of X_i. Since each X_i is connected, this separation must be trivial i.e. either one of $\widetilde{A}$ and $\widetilde{B}$ are empty. Since $x_0 \in \widetilde{A}$, $\widetilde{B}$ must be empty i.e. $X_i = A \cap X_i$ meaning that $X_i \subseteq A$. □

Therefore, $X = \bigcup_{i \in I} X_i \subseteq A$ implying that $B = \emptyset$ so X is connected. □

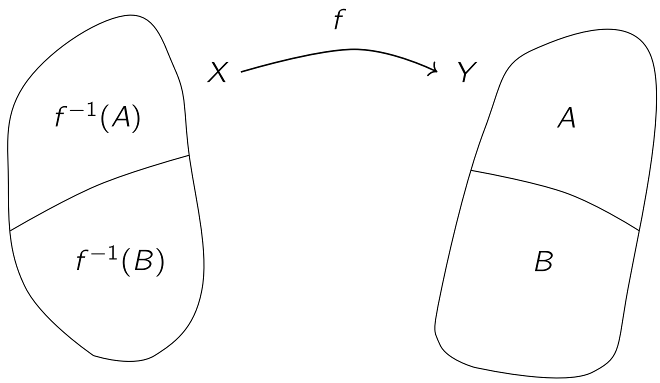
Remarks The collection is one of subspaces so that their union will remain a subset of X and we can equip it with a topology induced by X. Otherwise, it isn't clear how to construct a topology on the union. Furthermore, the conclusion holds even with the less restrictive condition of pairwise intersections being non-empty. The proof is an exercise.

Theorem 2.3 The continuous image of a connected set is connected.

Remarks

1) Connectedness is a topological invariant. It's useful to note that if given two spaces, one of which is connected and the other isn't, there doesn't exist a homeomorphism between them (i.e. they aren't homeomorphic to each other).
2) If X is an arbitrary connected topological space and $\sim$ is an equivalence relation on X, the quotient map $q\colon X \to X/\sim$ being continuous and surjective implies that $q(X) = X/\sim$ is connected.[1]

Proof. We can assume WLOG that $f\colon X \to Y$ is surjective (as we can simply restrict the range of f so that $f\colon X \to f(X)$). Let $Y = A \cup B$ be a non-trivial separation.



[1]Geometrically, the process of "gluing points" together in a connected space can't disconnect it.

Note that $X = f^{-1}(A) \cup f^{-1}(B)$ as every element of X is mapped into Y under f. Also, no element in X can be mapped simultaneously into A and B so $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint and so form a separation of X. Since f is surjective, $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty and thusly form a non-trivial separation of X. This contradicts the connectedness of X. Therefore, Y must be connected. □

2.2 Connected Subspaces of $\mathbb{R}$

Recall the intermediate value theorem from analysis i.e. given a continuous $f\colon [a, b] \to \mathbb{R}$, if $f(a) < f(b)$ then for every $f(a) < r < f(b)$ there exists a $c \in (a, b)$ such that $f(c) = r$. We wish to extend this property to the setting of general topological spaces. Thus, it's natural to consider the topological properties of $[a, b]$ on which the theorem depends. It turns out that the connectedness of $[a, b]$ fits the bill.

Theorem 2.4 (**Intermediate Value Theorem**) Let $f\colon X \to \mathbb{R}$ be a continuous map with X a connected topological space and $a, b \in X$ such that $f(a) \neq f(b)$. Let r be a number between $f(a)$ and $f(b)$. Then there exists a $c \in X$ such that $f(c) = r$.

Remarks

1) We were vague in saying that r is a number between $f(a)$ and $f(b)$. This just allows us more freedom to equally consider when $f(a) > f(b)$ and vice versa. There's no mystery here!

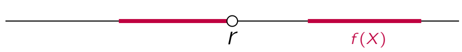
2) In general, we don't have a precise [2]way of saying that an element of a topological space lies "between" two others. However, $\mathbb{R}$ equipped with the standard topology is a special case of a linearly ordered set and so we can speak of $f(a) < r < f(b)$ for example.

3)i) Is the connectedness of X essential? Suppose that X isn't connected. Then there exists a non-trivial separation $X = A \cup B$. The map $f\colon X \to \{0, 1\}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in A \\ 1, & \text{if } x \in B \end{cases}$$

is continuous but doesn't obey the intermediate value property.

Proof. (IVT). Assume (for a contradiction) that such a c in the statement doesn't exist i.e. $r \notin f(X)$. Since X is connected, its image under f is also connected. As $f(X) \subseteq \mathbb{R}$, its elements either lie to the left or to the right of r, as illustrated below. Notice that we can write



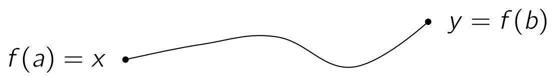
[2]For a well-ordered set X, we can endow X with a topology called the order topology. I don't have a neat resource for this right now. I'll update when I find one.

$$f(X) = \underbrace{((-\infty, r) \cap f(X))}_{A} \cup \underbrace{((r, \infty) \cap f(X))}_{B}$$

where A and B are open in $f(X)$ in the subspace topology. Therefore, they form a separation of $f(X)$. Since r is between $f(a)$ and $f(b)$, either $(f(a) \in A \wedge f(b) \in B)$ or $(f(a) \in B \wedge f(b) \in A)$. Thus, $f(X) = A \cup B$ is a non-trivial separation, contradicting the connectedness of $f(X)$. □

2.3 Path connectedness

Definition 2.5. Given any two points x, y in a topological space X, a path from x to y is a continuous map $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$.



We say that X is **path-connected** if for every pair of points $x, y \in X$, there exists a path between them contained entirely in X.

Remark Since $[a, b]$ and $[0, 1]$ are diffeomorphic via the map $[0, 1] \to [a, b]$, $t \mapsto a + t(b - a)$, we can equivalently define a path on $[0, 1]$.

Theorem 2.6 If X is a path-connected topological space, then it is connected.

Proof. Fix a point x_0 in X. Then for every $x \in X$, there exists a path $f_x : [a_x, b_x] \to X$ from x_0 to x. Denote the trace of the path by $C_x = f_x([a_x, b_x])$. Then $X = \bigcup_{x \in X} C_x$. Each C_x is connected as the continuous image of a connected space, namely $[a_x, b_x]$. All the C_x have a common point, x_0, so their union is also connected. □

e.g. The unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ is convex[3] and therefore path-connected via $f(t) = x + t(y - x)$.

e.g. For $n > 1$, $\mathbb{R}^n \setminus \{0\}$ is path connected. For any $x, y \in X$, take $f(t) = x + t(y - x)$. If the image of f happens to pass through the origin at some point, we can simply take another point $a \neq 0$ and concatenate[4] the paths from x to a and from a to y, thus avoiding the origin. This example extends to $\mathbb{R}^n \setminus \{x_1, \ldots, x_n\}$.

[3] The same holds true for any convex subspace of $\mathbb{R}^n$.

[4] For the time being, think of this as joining the two paths together. More details are to come later on when we talk about path connected components.

Proposition 2.7. The continuous image of a path-connected set is path-connected.

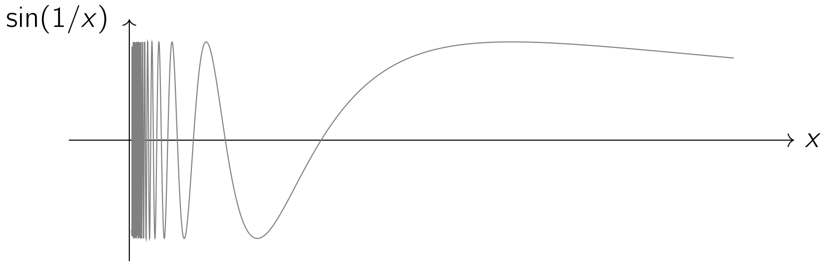
Proof. Let $h\colon X \to Y$ be a continuous map on a path-connected set X. If $x, y \in h(X)$, there exist $a, b \in X$ s.t. $h(a) = x$ and $h(b) = y$. Since X is path-connected, there exists a path $f : [\alpha, \beta] \to X$ with $f(\alpha) = a$ and $f(\beta) = b$. This implies that $h \circ f$ is a path from $x \to y$. Since x, y were arbitrary in $h(X)$, we conclude that $h(X)$ is path-connected. □

e.g. The map $f : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ defined by $x \mapsto x/\|x\|$ is continuous and from the prior example, $\mathbb{R}^n \setminus \{0\}$ is path-connected, so S^{n-1} is also path-connected.

2.4 Topologist's sine curve, $\overline{S}$

Now we consider a special set, $\overline{S}$, which demonstrates that path-connectedness in general is a stronger property than connectedness. This set is called the topologist's sine curve. We begin with:

$$S = \left\{\left(x, \sin\frac{1}{x}\right) : x \in (0, 1]\right\}$$



As the continuous image of a path-connected space (namely $(0, 1]$), S is also path-connected and therefore connected. Now consider the closure $\overline{S}$. The accumulation points of S lie in the set $\{(0, t) : t \in [-1, 1]\}$ so $\overline{S} = S \cup (\{0\} \times [-1, 1])$. If a space is connected, then so is its closure. (More generally, if A is connected and B is such that $A \subseteq B \subseteq \overline{A}$, then B is connected.) Thus, $\overline{S}$ is connected.

Claim $\overline{S}$ is not path-connected. More precisely, $(0, 0)$ can't be connected by a path to any point (x_0, y_0) in $\overline{S}$ with $x_0 > 0$.

Proof. Suppose that $f = (\gamma_1, \gamma_2)\colon [a, b] \to \overline{S}$ be such a path. Let $c_0 = \max \gamma_1^{-1}(\{0\})$ which is the maximum of a closed set that's contained in $[a, b]$ so c_0 exists. In other words, $\gamma_1(c_0) = 0$ and for all $t > c_0$, $\gamma_1(t) > 0$. Further, $c_0 < b$ and $\gamma_1(b) = x_0 > 0$. Since f is continuous and $[a, b]$ is connected, we can apply the IVT and deduce the existence of a sequence t_n such that $t_n \to c_0$ with

$$\gamma_1(t_n) = \frac{1}{\frac{\pi}{2} + n\pi} \qquad \gamma_2(t_n) = \sin\left(\frac{1}{\gamma_1(t_n)}\right) = \sin\left(\frac{\pi}{2} + n\pi\right).$$

Thus, we've found a sequence of points (x, y) with x arbitrarily close to 0 and y alternating between -1 and 1. This contradicts that the sequential limit $\gamma_2(t_n) \to \gamma_2(c_0)$ as $n \to \infty$. Therefore, $\overline{S}$ is not path-connected. □

2.5 Connected Components

If a topological space X is not connected, how can we detect which connected subset a point lies inside?

Definition 2.8. For any $x, y \in X$, define $\sim$ by $x \sim y$ if there exists a connected subspace $C \subseteq X$ that contains both x and y.

Is $\sim$ an equivalence relation?

Proof. Symmetry and reflexivity are clear. For transitivity, suppose that $x, y, z \in X$ and that $x \sim y$, $y \sim z$. This means that there exist connected subspaces C_1, C_2 of X such that $x, y \in C_1$ and $y, z \in C_2$. Both C_1 and C_2 have y as a common point so their union $C = C_1 \cup C_2$ is connected. Thus $x, z \in C$-connected so $x \sim z$. □

Now we can define the equivalence classes of X with respect to $\sim$. We call these equivalences classes the **connected components** of X. Thus, we may write X as the disjoint union $X = \bigcup_{i \in I} C_i$ where $\{C_i\}_{i \in I}$ denotes the collection of connected components of X. We call such a union a **connected component decomposition**.

e.g.

1) For a connected space X, its connected component decomposition is trivial, namely $X = X$.

2) Consider any set X equipped with the discrete topology on X, $(X, \mathcal{P}(X))$. The decomposition of X into connected components is $X = \bigcup_{x \in X} \{x\}$.

Is example 2 the only case of such a decomposition?

3) Consider $\mathbb{Q} \subseteq \mathbb{R}$ equipped with the subspace topology. The rationals are totally disconnected so the connected component decomposition is $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$. Note that the subspace topology is "very non-discrete".

Remark Indeed, this decomposition is possible if and only if the space is totally disconnected.

To justify the name of connected components, we present a theorem.

Theorem 2.9 The connected components of X are disjoint, connected subspaces of X such that every connected subspace $Y \subseteq X$ intersects at most one of these components.

Proof. Since the components of X are equivalence classes, they are disjoint and their union is X. Each connected subspace A of X intersects only one of them. If A intersected C_1 and C_2 at points x_1, x_2 respectively, then $x_1 \sim x_2$ and this can't happen unless $C_1 = C_2$. To show that each component C_i is connected, fix $x_0 \in C_i$. For each $x \in C_i$ there exists a connected space A_x containing x and x_0. Since $A_x \cap C_i \neq \emptyset$, $A_x \subseteq C_i$. Then the union $C_i = \bigcup_{x \in C_i} A_x$ is connected as all the A_x are connected and have x_0 as a point in common. □

e.g. Let $X = (0,1) \setminus \left\{\frac{1}{n} : n \in \mathbb{N}\right\} = \bigcup_{n \in N} \underbrace{\left(\frac{1}{n+1}, \frac{1}{n}\right)}_{A_n}$.

Claim The connected components of X are the A_n.

Proof. All the A_n are connected so for each A_n, there exists a connected component C such that $A_n \subseteq C$. Suppose that there exists a connected component C such that for $i \neq j$, $A_i \cup A_j \subseteq C$. Note that

$$\begin{aligned} A_i = \left(\frac{1}{i+1}, \frac{1}{i}\right) &= \left(\frac{1}{i+1}, \frac{1}{i}\right) \cap X \\ &= \left[\frac{1}{i+1}, \frac{1}{i}\right] \cap X \end{aligned}$$

so C contains a non-trivial clopen set and thus can't be connected. □

Proposition 2.10. The connected components C_i of $X = \bigcup_{i \in I} C_i$ are closed.

Proof. The closure $\overline{C_i}$ is connected and intersects at most one connected component of X, namely C_i. Thus, $\overline{C_i} \subseteq C_i$ so $\overline{C_i} = C_i$ and so the claim follows. □

Remark This statement in general is false for path-connected components.

2.6 Path-Connected Components

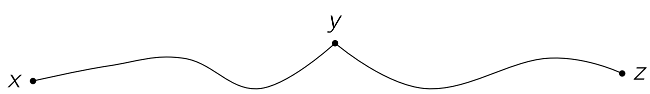
Definition 2.11. For $x, y \in X$, define $\sim$ by $x \sim y$ if there exists a path from x to y.

As before, we verify that $\sim$ is an equivalence relation.

Proof. Let $x, y, z \in X$. For reflexivity, simply take the constant path at x. For symmetry, given a path f from x to y, consider the reverse path $\widetilde{f}$ from y to x defined by $\widetilde{f}(t) = f(-t)$. For transitivity, let $x \sim y$ and $y \sim z$. Thus, there exist paths $f\colon [0,1] \to X$ and $g\colon [0,1] \to X$ from $x \rightsquigarrow y$ and $y \rightsquigarrow z$ respectively. We construct a map $h\colon [0,1] \to X$ defined by

$$h(s) = \begin{cases} f(2s), & s \in [0, 1/2] \\ g(2s-1), & s \in [1/2, 1]. \end{cases}$$

Both f and g are continuous maps that agree (with value $f(1/2) = g(1/2) = y$) on the intersection of the closed intervals $A = [0, 1/2]$ and $B = [1/2, 0]$. Therefore, the pasting lemma applies and h is a path from $h(0) = f(0) = x$ to $h(1) = g(1) = z$.



□

Once again, we can define the equivalence classes of X with respect to $\sim$ and refer to them as the path-connected components of X. The path-connected component decomposition of a space is defined analogously.

Now we have two types of decompositions of a topological space. Naturally, if possible, we'd like to relate the two somehow. We already know that if a space is path-connected, then it's necessarily connected. Thus, a path connected component decomposition is a refinement of its connected component counterpart.

e.g. Consider $\mathbb{Q} \subseteq \mathbb{R}$. Since $\mathbb{Q}$ is totally disconnected and we can't refine the connected component decomposition any further, it coincides with $\mathbb{Q}$'s path-connected component decomposition.

e.g. The space $X = (0,1) \subseteq \mathbb{R}$ is path connected so its path-connected component decomposition is trivial.

e.g. For the topologist's sine curve, $\overline{S}$, it's connected so has a trivial decomposition into connected components. How about its path-connected decomposition?

Claim $\overline{S}$'s decomposition into path connected components is

$$\overline{S} = A \cup B, \text{ where } A = S, \; B = \{0\} \times [-1,1].$$

Proof. Both A and B are path-connected so either $A \cup B$ is a single path-connected component or both A and B constitute distinct path-connected components. The former isn't possible because $A \cup B = \overline{S}$ isn't path-connected. □

e.g. $X = \overline{S} \setminus (\{0\} \times \mathbb{Q})$ is connected as S is connected and $S \subseteq X \subseteq \overline{S}$.

What are the path-connected components of X?

After removing the rational points $\{0\} \times [-1,1]$ from $\overline{S}$, S remains path-connected so it's contained in some path connected component of X (possibly larger than S itself). Had there been a path between $(0,i)$, where $i \in (\mathbb{R} \setminus \mathbb{Q}) \cap [-1,1]$, and any point in S, there would've been such a path in $\overline{S}$ before we removed any points. We've already ruled out such a case so S is a path-connected component of X.

Claim Every point $(0,i)$ belongs to a different path-connected component.

Proof. Suppose that between any two points $(0,i_1)$ and $(0,i_2)$, there exists a path p between them. The path must be contained in $X \setminus S$ otherwise $(0,i_1)$ would be in the same path-connected component as S. Since p is continuous, by the IVT p must attain a point on $\{0\} \times [-1,1]$ with a rational second coordinate. This is a contradiction. □

Therefore, the path-connected component decomposition of X is

$$X = S \bigcup \{(0,i)\}_{i \in (\mathbb{R} \setminus \mathbb{Q}) \cap [-1,1]}.$$

Remark Note that removing S from X would totally disconnect it.

2.7 Compact Spaces

Recall that if $A \subseteq \mathbb{R}^n$ is compact and $f\colon A \to \mathbb{R}$ is continuous, then f attains both extrema and is uniformly continuous. We'd like to find a topological generalisation of this notion of compactness we had in Euclidean spaces.

Definition 2.12. Let X be a topological space and $\mathcal{A} = \{A_i\}_{i\in I}$ be a family of subsets of X.

1) We call $\mathcal{A}$ a **cover** of X if $X = \bigcup_{i\in I} A_i$.
2) If $\mathcal{A}$ is a cover of X and every element $A_i \in \mathcal{A}$ is open, we call $\mathcal{A}$ an **open cover** of X.
3) A **finite subcover** of a cover $\mathcal{A}$ is a subcollection $\{A_i\}_{i\in I'} \subseteq \mathcal{A}$ where $|I'| < \infty$ and $X = \bigcup_{i\in I'} A_i$.
4) We call X **compact** if every open cover $\mathcal{A}$ of X admits a finite subcover of X.

e.g. The real line $\mathbb{R}$ is not compact. It suffices to find a single open cover of $\mathbb{R}$ that doesn't admit a finite subcover. Consider the open cover $\mathcal{A} = \{(z-1, z+1) : z \in \mathbb{Z}\}$. Any finite subcollection, say $\{(z-1, z+1) : z \in I'\} \subseteq \mathcal{A}$, where $|I'| < \infty$, will have a maximum $Z = \max\{|z| : z \in I'\}$. Now note that

$$\bigcup_{z\in I'} (z-1, z+1) \subseteq (Z-1, Z+1) \subsetneq \mathbb{R}.$$

The claim follows.

e.g. $X = (0, 1) \cong \mathbb{R}$ is not compact. If our topological definition of compactness is to be consistent[5] with the Euclidean definition (closed and bounded), then $(0, 1)$ better not be compact. We verify this by explicitly constructing an open cover that admits no finite subcover:

$$X = \bigcup_{n\in\mathbb{N}} \left(\frac{1}{n+2}, \frac{1}{n}\right)$$

Suppose that there exists a finite subcover. This subcollection will have a positive minimum value and we can always find an $x \in (0, 1)$ such that x is smaller than this minimum. Thus, the claim follows.

e.g. $X = [0, 1]$

Claim X is compact.

Proof. Let $\mathcal{A} = \{A_i\}_{i\in I}$ be an open cover of X. Call $X = X_0$ and divide it in half i.e. $X_0 = [0, 1/2] \cup [1/2, 1]$ and suppose that X_0 can't be covered by finitely many of the A_i. Then, we can't cover either $[0, 1/2]$ or $[1/2, 1]$ by finitely many of the A_i. Call the interval that can't be covered in such a way, X_1. Proceeding inductively, we obtain a sequence $\{X_i\}_{i\in\mathbb{N}_0}$ of intervals with length 2^{-i} that can't be covered by finitely many A_i. This sequence is nested i.e. for each $i \in \mathbb{N}_0$, $X_{i+1} \subseteq X_i$.

[5] We'll also see that compactness is a topological invariant so $\mathbb{R}$ and $(0, 1)$ being homeomorphic and the real line not being compact necessarily implies that $(0, 1)$, too, is not compact.

Lemma (**Cantor's Lemma**) For a nested sequence $\{X_i\}_{i\in\mathbb{N}}$ of closed and bounded sets in $\mathbb{R}$, their intersection is non-empty and $\bigcap_{i\in\mathbb{N}} X_i = \{x_0\}$. Equivalently, if $X_i = [x_{i_1}, x_{i_2}]$, then $\{x_{i_1}\}_{i\in\mathbb{N}}, \{x_{i_2}\}_{i\in\mathbb{N}}$ are convergent with the same limit x_0 as $i \to \infty$.

Since $\{A_i\}_{i\in I}$ is an open cover of X_i, there exists an $i_0 \in I$ such that $x_0 \in A_{i_0}$. Since A_{i_0} is open, there exists an $\varepsilon_0 > 0$ so that for sufficiently large i, $X_i \subseteq (x - \varepsilon_0, x + \varepsilon_0) \subseteq A_{i_0}$. This contradicts that X_i can't be covered by finitely many A_i. Indeed we covered X_i by a single open set A_i. □

e.g. $S = \left\{\dfrac{1}{n} : n \in \mathbb{N}\right\} \subseteq \mathbb{R}$

By the Euclidean definition, S is bounded but not closed (since 0 is a limit point but not in the closure). Each singleton set is open in the subspace topology on S so take S as its own open cover. This set is infinite so any finite subcollection won't cover S. Thus, S isn't compact.

Remark The topology here in the example above is discrete and $(X, \mathcal{P})$ is compact iff $|X| < \infty$.

e.g. Consider $X = S \cup \{0\}$. Given an open cover $X = \bigcup_{i\in I} A_i$ of X, there exists at least one $i_0 \in I$ such that A_{i_0} contains the origin. As A_{i_0} is open, it in particular contains a neighbourhood of the origin i.e. all but finitely many elements

$$\left\{\frac{1}{n_1}, \ldots, \frac{1}{n_k}\right\} \subseteq S.$$

For all $j \leqslant k$, there exists an $i_j \in I$ such that $1/n_j \in A_{i_j}$. Then $X = A_{i_0} \cup A_{i_1} \cup \cdots \cup A_{i_k}$ is an finite subcover of our original arbitrary open cover. Thus X is compact.

e.g. Let X be an infinite set equipped with the finite complement topology, $\mathcal{T}_f$. If $X = \bigcup_{i\in I} A_i$ is an open covering of X, there exists an $i_0 \in I$ such that $|X \setminus A_{i_0}| < \infty$ i.e. $X \setminus A_{i_0} = \{x_1, \ldots, x_k\}$. For every $j \leqslant k$, there exists an $i_j \in I$ such that $x_j \in A_{i_j}$. Therefore, $X = A_{i_0} \cup A_{i_1} \cup \cdots \cup A_{i_k}$ is a finite subcover.

Remark For any $Y \subseteq (X, \mathcal{T}_f)$, the induced topology will also be a finite complement topology so Y is also compact.

e.g. We've already found that $\mathbb{R}$ isn't compact. It turns out that we can compactify[6] the real line by adjoining a single point (to form a set $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$) and constructing a topology $\mathcal{O}$ on this set which is compact and such that $\mathcal{O}|_{\mathbb{R}}$ coincides with the standard topology on $\mathbb{R}$.

Suppose that X is a topological space and $Y \subseteq X$ is a subspace. Given a topology on X, we'd like to discuss whether Y is compact. It turns out that an intrinsic open covering of Y (i.e. by sets in Y) and an extrinsic open covering of Y (by sets open in X) are equivalent notions.

Let $Y = \bigcup_{i\in I} A_i$ be an open covering of Y. For each $i \in I$, there exists a B_i open in X such that $A_i = B_i = Y$. Thus $Y = \bigcup_{i\in I} B_i \cap Y$ i.e. $Y \subseteq \bigcup_{i\in I} B_i$.

Conversely, given a family of open sets $\{B_i\}_{i\in I}$ in X satisfying $Y \subseteq \bigcup_{i\in I} B_i$, we can write $Y = (\bigcup_{i\in I} B_i) \cap Y = \bigcup_{i\in I} \underbrace{(B_i \cap Y)}_{=:A_i}$ i.e. $\{A_i\}_{i\in I}$ is an intrinsic open covering of Y.

[6] This is called the Alexander/one-point compactification of a space.

This example seems innocent enough but it happens to be a very powerful change of viewpoint when it comes to proving facts about subspaces!

Theorem 2.13 Let X be a compact topological space and $Y \subseteq X$ be closed in X. Then Y is compact.

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of Y by open sets U_i in X. Then $X = (\bigcup_{i \in I} U_i) \cup (X \setminus Y)$ is an open cover of X. Since X is compact, this open cover admits a finite subcover i.e. $\exists I' \subseteq I$ with $|I'| < \infty$ such that $X = \bigcup_{i \in I'} U_i$. This is certainly a finite open cover of Y. If $X \setminus Y$ is one of the U_i, it contains no points of Y so we may certainly throw it away, leaving a finite subcover. Otherwise, we leave $\{U_i\}_{i \in I'}$ alone. Either way, we conclude that Y is compact. □

Is the converse statement true i.e. is a compact subspace $Y \subseteq X$ necessarily closed?
Recall that every subspace (closed or not) of $(X, \mathcal{T}_f)$, where X is an infinite set, is compact.

The next theorem is another step on the way to proving that in a Euclidean space (which are all Hausdorff), closed and boundedness are equivalent to being compact.

Theorem 2.14 Let X be a Hausdorff topological space and $Y \subseteq X$ be a compact subspace. Then Y is closed in X.

Proof. We equivalently show that $X \setminus Y$ is open. Fix $x_0 \in X \setminus Y$. We aim to excise a neighbourhood of x_0 contained in $X \setminus Y$ and x_0 being arbitrary'd mean that $X \setminus Y$ is open.

As X is Hausdorff, for each $y \in Y$ there exist disjoint open neighbourhoods[7] U_y, V_y in X of y, x_0 respectively. The family $\{U_y\}_{y \in Y}$ is an open cover of Y (by sets open in X) so $Y \subseteq \bigcup_{y \in Y} U_y$. By the compactness of Y, there exists a finite subcover $\{U_{y_1}, \ldots, U_{y_n}\}$ of Y. Set $V = \bigcap_{i=1}^n V_{y_i}$ to be an open neighbourhood of x_0. We claim that $V \cap Y = \emptyset$ i.e. that this is the neighbourhood that we wanted in $X \setminus Y$. Observe that:

$$\begin{aligned} V \cap Y \subseteq V \cap \left(\bigcup_{i=1}^n U_{y_i} \right) &= \bigcup_{i=1}^n \left(U_{y_i} \cap V \right) \\ &= \bigcup_{i=1}^n \left(U_{y_i} \cap \left(\bigcap_{j=1}^n V_{y_j} \right) \right) \\ &\subseteq \bigcup_{i=1}^n \left(U_{y_i} \cap V_{y_i} \right) = \bigcup_{i=1}^n \emptyset = \emptyset \end{aligned}$$

□

[7] A priori, the neighbourhoods depend on x_0 and y but we've fixed the former so we'll suppress its dependence.

e.g.

1) Consider $[a, b] \subseteq \mathbb{R}$. Since $\mathbb{R}$ is Hausdorff and $[a, b]$ is closed, $[a, b]$ is compact. This is consistent with our proof that $[0, 1]$ is compact since $[a, b] \cong [0, 1]$.

2) $[a, b) \cong (a, b]$ aren't compact as they aren't closed subsets of the Hausdorff space $\mathbb{R}$.

Theorem 2.15 The continuous image of a compact space is compact.

Proof. Let $f : X \to Y$ be continuous and X be compact. WLOG, we can restrict f onto its image and suppose that f is surjective. Let $\{A_i\}_{i\in I}$ be an open cover of Y. As f is surjective, $\{f^{-1}(A_i)\}_{i\in I}$ is an open cover of X. By the compactness of X, this open cover admits a finite subcovering $\{f^{-1}(A_i)\}_{i\in I'}$ with $I' \subseteq I$ and $|I'| < \infty$. Since f is surjective[8], $f(f^{-1}(A_i)) = A_i$ for each $i \in I'$. Thus $\{A_i\}_{i\in I'}$ is a finite open cover of Y. Thus, Y is compact. □

e.g.

1) Any quotient space of a compact space is also compact.

2) The trace $f([a, b])$ of any path $f : [a, b] \to X$ is compact.

e.g. Suppose that X is a product space $X = X_1 \times X_2$ and that X is compact. Since π_1, π_2 are surjective continuous maps, both X_1 and X_2 are compact. This can be extended to a product of arbitrarily many spaces. We can also conclude that for some $i_0 \in I$, the image of the map that excludes exactly one element of the product

$$\widetilde{\pi}_{i_0} : X \to \prod_{i\in I\setminus\{i_0\}} X_i$$

is compact.

Theorem 2.16 (Finite Compact Product) The product of finitely many compact spaces is compact.

e.g. $[0, 1]^n \subseteq \mathbb{R}^n$ is compact.

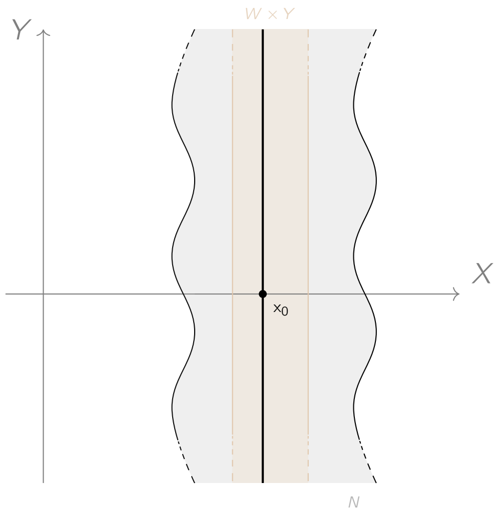
Theorem 2.17 (**Tychonoff's Theorem**) If $\{X_i\}_{i\in I}$ is an arbitrary collection of compact topological spaces, then their product equipped with the product topology is compact.

e.g. $[a, b]^\omega = \prod_{n\in N} [a, b]$ is compact in the product topology. With the box topology, $[a, b]^\omega$ isn't a[9] compact.

[8]For a general $f : X \to Y$ with $A \subseteq Y$, we have that $f(f^{-1}(A)) \subseteq A$.
[9]In other literature, compact spaces can be referred to as compacts.

In order to prove the Finite Compact Product theorem, we'll need a preparatory lemma.

Tube Lemma Let X, Y be topological spaces, Y-compact, $x_0 \in X$ fixed and $N \subseteq X \times Y$ be an open set in the product topology containing the slice $\{x_0\} \times Y \subseteq N$. Then there exists a tube i.e. a neighbourhood W of x_0 in X such that $N \supseteq W \times Y$.



Proof of the Finite Compact Product Theorem. Assume that X, Y are compact topological spaces. Assuming the tube lemma, we'll prove that their product is compact. By induction, we'll extend to a finite product.

Suppose that we have an open cover $\{A_i\}_{i\in I}$ of $X \times Y$. Let $x_0 \in X$ be an arbitrary point and consider the slice $\{x_0\} \times Y \subseteq \bigcup_{i\in I} A_i$. As $\{x_0\} \times Y \cong Y$ and Y is compact, there exists a finite indexing set $I' \subseteq I$ with

$$\{x_0\} \times Y \subseteq \left(\bigcup_{i\in I'} A_i\right) =: N.$$

By the tube lemma, N contains a tube i.e. there exists a neighbourhood W of x_0 such that $W \times Y \subseteq N = \bigcup_{i\in I'} A_i$. So far, we've proven that for[10] every $x \in X$, there exists a neighbourhood W_x of x and a finite indexing set $I'_x \subseteq I$ such that $W_x \times Y \subseteq \bigcup_{i\in I'_x} A_i$. Since $\{W_x\}_{x\in X}$ is an open cover of X-compact, there exists a finite set $\{x_1, \dots, x_n\} \subseteq X$ such that $X = \bigcup_{i=1}^n W_{x_i}$. Then

$$X \times Y = \left(\bigcup_{i=1}^n W_{x_i}\right) \times Y = \bigcup_{i=1}^n \left(W_{x_i} \times Y\right) \subseteq \bigcup_{i=1}^n \left(\bigcup_{j\in I'_{x_i}} A_j\right)$$

the last of which is a finite union of a finite union so this constitutes a finite subcovering. ◻

[10] The problem is that there are potentially infinitely many $x \in X$. By the compactness of X, we can narrow this down.

Proof of the Tube Lemma. Consider $\mathcal{A} = \{U \times V \subseteq N : U \subseteq X,\ V \subseteq Y \text{ are open and } x_0 \in U\}$. This is an open cover of $\{x_0\} \times Y$ (by the definition of the product topology). Since $\{x_0\} \times Y \cong Y$ and Y is compact, $\{x_0\} \times Y$ is also compact so $\mathcal{A}$ admits a finite subcover $\{U_i \times V_i\}_{i=1}^n$ of $\{x_0\} \times Y$. Take $W = \bigcap_{i=1}^n U_i$ i.e. an open neighbourhood of x_0. For every $(x, y) \in W \times Y$, there exists an $i_0 \in \{1, \ldots, n\}$ such that $y \in V_{i_0}$ and $x \in W \subseteq U_{i_0}$. Therefore, $(x, y) \in U_{i_0} \times V_{i_0} \subseteq N$ (by the definition of $\mathcal{A}$). □

e.g.

1) $[0, 1]^n \subseteq \mathbb{R}^n$ is compact.

2) $[0, 1]^\omega \subseteq \mathbb{R}^\omega$ is compact by Tychonoff's theorem.

3) $[0, 1]^\omega \subseteq \mathbb{R}^\omega$ is not compact when equipped with the box topology.

2.7.1 Compact Subspaces of $\mathbb{R}$

Recall that a closed subspace of a compact space is compact and that if Y is a compact subspace of a Hausdorff space, then Y is closed.

Theorem 2.18 Let $A \subseteq \mathbb{R}^n$. Then A is compact iff A is closed[11] and bounded.[12]

Proof. Suppose that $A \subseteq \mathbb{R}^n$ is compact. Since $\mathbb{R}^n$ is Hausdorff, A is closed. Any compact space is bounded in every metric generating the topology on the space (from an assignment). For the converse, A being bounded means that $\exists R > 0$ such that $A \subseteq [-R, R]^n$ where $[-R, R]^n$ is compact as a finite product of compact spaces. As a closed subspace of a compact space, A is therefore compact. □

Theorem 2.19 (**Extreme Value Theorem**) Let X be a compact topological space and $f : X \to \mathbb{R}$ be a continuous map. Then f attains its extrema i.e. $\exists c, d \in X$ s.t. for all $x \in X$, $f(c) \leqslant f(x) \leqslant f(d)$.

Proof. WLOG, we may assume that $f : X \to f(X)$ is surjective. We're going to prove that $f(X)$ has a maximum. Suppose that such a maximum doesn't exist. Since $f(X)$ is the continuous image of a compact space, it is also compact. Since $f(X)$ is unbounded above, for every $a \in f(X)$ $\exists b \in f(X)$ s.t. $b > a \implies a \in (-\infty, b)$ so we may re-write $f(X)$ as:

$$f(X) = f(X) \cap \mathbb{R} = f(X) \cap \left(\bigcup_{b \in f(X)} (-\infty, b) \right) = \underbrace{\bigcup_{b \in f(X)} (f(X) \cap (-\infty, b))}_{\vdots}.$$

This is an open cover of $f(X)$ by sets open in $f(X)$. As $f(X)$ is compact, there exists a finite subcover i.e. $\exists b_1, \ldots, b_k \in f(X)$ such that

$$f(X) \subseteq \bigcup_{i=1}^k (-\infty, b_i).$$

[11] Topologically i.e. the complement of an open set.
[12] With respect to the Euclidean metric.

WLOG, assume that $b_k = \max_{1\leqslant i\leqslant k}\{b_i\}$ so that $b_k \in f(X) \subseteq (-\infty, b_k)$. This is a contradiction. Thus, there is such a maximum. The argument for a minimum is similar. Thus, f attains its extrema. □

Recall Cantor's theorem which states that a continuous real-valued function on $[a, b]$ is uniformly continuous. We can generalise this to the setting of metric spaces. First come a few definitions and the concept of a Lebesgue number.

Definition 2.20. Let (X, d_X) and (Y, d_Y) be metric spaces.

A function $f: (X, d_X) \to (Y, d_Y)$ is called **continuous at** $x_0 \in X$ if for every $\varepsilon > 0$, there exists a $\delta_{\varepsilon,x_0} > 0$ such that $d_X(x_0, x) < \delta_{\varepsilon,x_0} \implies d_Y(f(x_0), f(x)) < \varepsilon$.

A function $f: (X, d_X) \to (Y, d_Y)$ is called **uniformly continuous on** X if for every $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that for all $x, y \in X$, $d_X(x, y) < \delta_\varepsilon \implies d_Y(f(x), f(y)) < \varepsilon$.

Definition 2.21. Let (X, d) be a metric space and $A \subseteq X$. The diameter of A is defined to be

$$\operatorname{diam}(A) := \sup_{x,y\in A} d(x, y).$$

Lemma 6 (**Lebesgue Number**) Let (X, d) be a compact metric space and $\mathcal{A} = \{A_i\}_{i\in I}$ be an open cover of X. Then $\exists\delta > 0$ such that for all $A \subseteq X$ with $\operatorname{diam}(A) < \delta$, there exists an $i_0 \in I$ such that $A \subseteq A_{i_0}$.

Proof. Omitted.[13] □

e.g. We'll consider an open covering of $X = \mathbb{R}$ that doesn't admit a Lebesgue number:

$$X = \left(\bigcup_{n\in\mathbb{Z}}(n, n+1)\right) \cup \underbrace{\left(\bigcup_{n\in\mathbb{Z}} \mathbb{B}_{\frac{1}{|n|+1}}(n)\right)}_{\vdots}.$$

(For the radius of the balls, we could've taken any function $\overset{n\to\infty}{\longrightarrow} 0$.)

Given $\delta > 0$, take $U_n = (n - \delta/3, n + \delta/3)$. This is an open set whose diameter is $2\delta/3 < \delta$ and for $n >> 0$, U_n is not contained in any of the members of the covering.

Remark It's not the case that for a space that isn't compact, every open cover doesn't admit a Lebesgue number. Indeed, some open covers of $\mathbb{R}$ do admit Lebesgue numbers (possibly infinite in some cases).

[13] A plain English statement of this theorem is as follows: for every subset of a compact space with diameter less than a Lebesgue number, there's at least one element of the cover that contains it.

Theorem 2.22 Let $f\colon (X, d_X) \to (Y, d_Y)$ be a continuous map. If X is compact, then f is uniformly continuous.

Proof. Let $f\colon X \to Y$ be continuous. We may suppose that f is surjective onto its image. Let $\varepsilon > 0$. We're going to construct a δ_ε. Consider the set of metric balls $\mathcal{A}' = \{\mathbb{B}_{d_Y}(y, \varepsilon/2) : y \in Y\}$ which is an open covering of Y. Take $\mathcal{A} = \{f^{-1}(\mathbb{B}_{d_Y}(y, \varepsilon/2)) : y \in Y\}$. This is an open cover of X by the surjectivity of f. As X is assumed to be compact, it has a Lebesgue number $\delta > 0$.

Claim This is the δ_ε we've been looking for.

Suppose that $d_X(x, \widetilde{x}) < \delta$ i.e. $\operatorname{diam}(\{x, \widetilde{x}\}) < \delta$. By the definition of a Lebesgue number, there exists a $y \in Y$ such that $\{x, \widetilde{x}\} \subseteq f^{-1}(\mathbb{B}(y, \varepsilon/2))$. This implies that $f(x), f(\widetilde{x}) \in \mathbb{B}(y, \varepsilon/2)$. By the triangle inequality, we may conclude that

$$d(f(x), f(\widetilde{x})) \leqslant d(f(x), f(y)) + d(f(y), f(\widetilde{x})) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

□

2.8 Local Compactness

Definition 2.23. A topological space X is called **locally compact** if for every $x \in X$, there exists a compact set $C \subseteq X$ that contains x and a neighbourhood of x.

e.g.

1) If X is compact, then X is locally compact (let $X = C$).

2) $X = \mathbb{R}$ is locally compact. Given $x \in \mathbb{R}$, simply take $C = [x-1, x+1]$.

3) $X = \mathbb{Q} \subseteq \mathbb{R}$ isn't compact.

   Claim It isn't locally compact either!

   Proof. Take $0 \in \mathbb{Q}$. **Subclaim** No set C containing 0 and a neighbourhood of 0 is compact.

   Suppose that such a set exists i.e. $C \supseteq ((-\varepsilon, \varepsilon) \cap \mathbb{Q})$ is compact. Consider $[\alpha, \beta] \subseteq (-\varepsilon, \varepsilon)$ where $\alpha, \beta \in \mathbb{R}$ with $|\alpha|, |\beta| < \varepsilon$. Then $C \supseteq ((\alpha, \beta) \cap \mathbb{Q}) =: A$ is closed in a compact space, so A is compact. Since A is compact in $\mathbb{R}$ which is a Hausdorff space, A is closed in $\mathbb{R}$. Thus, A is equal to it's own closure in $\mathbb{R}$. However, this contradicts that

   $$\overline{A}_{\mathbb{R}} = [\alpha, \beta] \supsetneq [\alpha, \beta] \cap \mathbb{Q} = A$$

   □

4) $X = \mathbb{R}^\omega$ equipped with the product topology isn't locally compact. Suppose that C is a compact subset of X containing any open set i.e.

$$C \supseteq \prod_{i \neq i_1, \ldots, i_k} \mathbb{R} \times \prod_{j=1}^{k} U_{i_j}$$

Take any $i \notin \{i_1, \ldots, i_k\}$ and consider the projection $\pi_i(C) = \mathbb{R}$. This contradicts the continuous image of a compact set being compact. Thus, the claim follows.

5) Note that the topologist's sine curve $S = \{(x, \sin 1/x) : x \in (0, 1]\} \cong (0, 1]$. Given $(x, \sin 1/x) \in S$, take $[a, b] \subseteq (0, 1]$ with $x \in [a, b]$ so that the image of $[a, b]$ under $f : x \mapsto (x, \sin 1/x)$ (a continuous map) is a compact set. Thus, S is locally compact.

Chapter 3

Algebraic Topology

For a thorough treatment of the fundamental group of a topological space and the notions of homotopy (equivalence), retractions and covering spaces, I highly recommend reading the 1st chapter from [3].[1] What follows is a mixture of the lectures and Hatcher's text.

The main goal of algebraic topology is to classify topological spaces up to homeomorphism through their algebraic structures. Thus we're interested in techniques for forming algebraic images of topological spaces. The mechanisms that create these images are called functors and have the characteristic feature that they form images of not only spaces, but also of maps. Thus, continuous maps between spaces are projected onto homomorphisms between their algebraic images.

One of the simplest and most important functors of algebraic topology is called the fundamental group. It creates an algebraic image of a space from the loops in the space. The fundamental group is defined in terms of loops and deformations of loops.

3.1 Homotopy

Definition 3.1. Let X and Y be topological spaces and $f_0, f_1 \colon X \to Y$ be maps. We say that f_0 is **homotopic** to f_1 if there exists a homotopy between them. A **homotopy** is a family of maps $f_t \colon X \to Y$, $t \in I = [0,1] \subseteq \mathbb{R}$, such that the associated map $F \colon X \times I \to Y$ given by $F(x,t) = f_t(x)$ is continuous in the product topology.

In particular, we may consider paths with fixed endpoints. Thus, the idea of continuously deforming a path while keeping its endpoints fixed is made precise by homotopies.

Definition 3.2. A **path** in a space X is a continuous map $f \colon I \to X$. A **homotopy of paths** in X a family of maps $f_t \colon I \to X$, where $t \in I$, such that the endpoints $f_t(0) = x_0$, $f_t(1) = x_1$ are independent of t and the associated map $F \colon I \times I \to X$ defined by $F(s,t) = f_t(s)$ is continuous. When two paths f_0, f_1 are connected via a homotopy f_t, they are said to be **path homotopic** and we write $f_0 \simeq f_1$.

[1] https://www.math.cornell.edu/~hatcher/AT/AT.pdf

Proposition 3.3. The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation. (Thus, we may define the equivalence class of a path under the relation of path homotopy. We denote the equivalence class $[f]$ and call it the **homotopy class of** f.)

Proof. Reflexivity is evident via the constant homotopy $f_t = f$ for all $t \in I$. For symmetry, if $f_0 \simeq f_1$ via the homotopy f_t, then $f_1 \simeq f_0$ via the homotopy f_{1-t}. For transitivity, suppose that $f_0 \simeq f_1$ via the homotopy f_t and that $f_1 = g_0$ with $g_0 \simeq g_1$ via the homotopy g_t. Then $f_0 \simeq g_1$ via the homotopy h_t, where $h_t = f_{2t}$ over $t \in [0, 1/2]$ and $h_t = g_{2t-1}$ over $t \in [1/2, 1]$. These two definitions agree for $t = 1/2$ as we've assumed that $f_1 = g_0$. The associated map $H(s, t) = h_t(s)$ is given by

$$H(s,t) = \begin{cases} F(s, 2t), & (s,t) \in I \times [0, 1/2] \\ G(s, 2t-1), & (s,t) \in I \times [1/2, 1] \end{cases}$$

and is continuous on $I \times I$ by the pasting lemma. □

Definition 3.4. Given two paths $f, g \colon I \to X$ such that $f(1) = g(0)$, there is a **composition** or **product path** $f \cdot g$ that traverses f followed by g and is defined by

$$(f \cdot g)(s) = \begin{cases} f(2s), & s \in [0, 1/2] \\ g(2s-1), & s \in [1/2, 1]. \end{cases}$$

Note that f and g are traversed twice as fast so that their product is a map from the unit interval into X.

It turns out that path concatenation respects homotopy classes i.e. $[f][g] = [f \cdot g]$. This is because if $f_0 \simeq f_1$ and $g_0 \simeq g_1$ via the homotopies f_t, g_t respectively with $f_0(1) = g_0(0)$ (so that $f_0 \cdot g_0$ is defined), then $f_t \cdot g_t$ is defined and provides a homotopy $f_0 \cdot g_0 \simeq f_1 \cdot g_1$.

3.2 The Fundamental Group (π_1)

In particular, suppose that we restrict our attention to paths $f \colon I \to X$ with the same starting and endpoint, $f(0) = f(1) = x_0 \in X$. Such paths are called **loops** and x_0 is called the basepoint of the loop. The set of all homotopy classes $[f]$ of loops $f \colon I \to X$ at the basepoint x_0 is denoted $\pi_1(X, x_0)$.

Theorem 3.5 The set $\pi_1(X, x_0)$ equipped with the operation $[f][g] = [f \cdot g]$ for all loops f, g in X is a group. We call this group the **fundamental group of** X **at the basepoint** x_0.

Now it's natural to ask how the fundamental group of a space X depends on its basepoint $x_0 \in X$. Since $\pi_1(X, x_0)$ involves only the path-connected component of X containing x_0, we can only hope to find a relationship between fundamental groups if their basepoints lie in the same path-connected component of X.

Definition 3.6. For a path $\gamma \colon I \to X$ from x_1 to x_0, we define a **change of basepoint map** $\beta_\gamma \colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ by $\beta_\gamma([f]) = [\gamma \cdot f \cdot \overline{\gamma}]$ where $\overline{\gamma}$ denotes the reverse path from x_0 to x_1.

Remarks

- Note that we have a choice of paths $(\gamma \cdot f) \cdot \overline{\gamma}$ and $\gamma \cdot (f \cdot \overline{\gamma})$ but they're homotopic and we're only only interested in homotopy classes so either choice suffices for the purpose of our map β_γ.
- The change-of-basepoint map is well defined as f_t being a homotopy of loops based at x_1 implies that $\gamma \cdot f \cdot \overline{\gamma}$ is a homotopy of loops based at x_0.

Proposition 3.7. The change of basepoint map β_γ is an isomorphism.

Thus, for a path-connected space X, the group $\pi_1(X, x_0)$ is, up to isomorphism, independent of the choice of basepoint x_0. In this case, we just abbreviate $\pi_1(X, x_0)$ to $\pi_1(X)$.

Definition 3.8. A topological space X is called **simply connected** if it is path connected and has a trivial fundamental group. What this means is that all loops in the space are homotopic to a constant loop. Such loops are called contractible.

This last chapter was somewhat rushed and I added details to fill the blanks in. Thus, it'd be wise to consult your own notes as for what's examinable. The fundamental group is actually the first in a sequence of groups $\pi_n(X, x_0)$ called homotopy groups. We won't go into homotopy groups here for $n \geqslant 2$.

Bibliography

[1] Munkres, James R. *Topology, a first course*. Prentice-Hall Inc., Englewood Cliffs, 1975.

[2] Sutherland, Wilson A. *An Introduction to Metric & Topological Spaces (2nd Edition)*. Oxford University Press, Oxford, 2008.

[3] Hatcher, Allen. *Algebraic Topology*. Cambridge University Press, Cambridge, 2002.

Congrats on reaching the end and I wish you all the best of luck in the exam.

- Khallil ♥