Linear Alkebra

by Khallil Ebrahim Benyattou

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Motivation

As part of my machine learning studies in 102, I worked through the basic theory of multiple linear regression. A particular type of matrix $\mathcal{X}^\top\mathcal{X}$ appeared in the normal equations that can be solved to obtain an optimal solution to a loss minimisation problem (a least-squares minimiser). Such matrices are called Gram matrices.

The Gram matrix $\mathcal{X}^\top\mathcal{X}$ from the normal equations is real, symmetric and positive semidefinite. It's a fact of linear algebra that real, symmetric matrices are equivalently characterised by admission of an orthogonal spectral decomposition.[1]

It was at this point that I realised my knowledge of linear algebra was very shaky — I'd forgotten a lot. For the time being, I'm thinking of an orthogonal matrix as the matrix representation of a linear operator that preserves the geometric concepts of "angles" and "lengths" in space.

The purpose of these notes is to document my work through the basics of linear algebra **until I reach a point** at which I can prove this theorem.

I did end up getting **this far**! **Section 3.3.3** is the natural stopping point to understand this orthogonal diagonalisability. I then discuss the Gram matrix in the normal equations in **Section 6.4**.

I also decided to go a little further and study orthogonal projections, spectral decomposition, polar decomposition, and the singular value theorem.

[1] A spectral decomposition is a special case of eigendecomposition when the matrix is real and symmetric (or normal). An **eigendecomposition** is a factorisation of a matrix into a canonical form, whereby the matrix is represented in terms of its eigenvalues and eigenvectors. Only diagonalisable matrices can be factorised in this way. Quote taken straight from the 'Eigendecomposition of a matrix' page on Wikipedia.

CHAPTER 1

Linear Algebra

Let V be a vector space over a field $\mathbb{K}$.

1.1 Linear Independence and Bases

Definition 1.1.1 Let $\mathbf{v}_1, \ldots, \mathbf{v}_r \in V$ be a collection of vectors in V.

- The collection is called **linearly dependent** if there exist scalars $\alpha_1, \ldots, \alpha_r \in \mathbb{K}$, *not all zero*, such that
$$\alpha_1 \mathbf{v}_1 + \ldots + \alpha_r \mathbf{v}_r = \mathbf{0}_V.$$
- If the vectors aren't linearly *dependent*, we say that they are **linearly *independent*** i.e. the only scalars for which the above equation holds are $\alpha_1 = \ldots = \alpha_r = 0_{\mathbb{K}}$.

  Lemma 1.1.2 The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly dependent iff either $\mathbf{v}_1 = \mathbf{0}_V$, or, for some j, $\mathbf{v}_j$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

- The collection is said to **span V** if every $v \in V$ may be written as a linear combination of the $\mathbf{v}_i$.

A vector space is called **finite-dimensional** if it contains a finite spanning set. A collection $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is called a **basis for V** if it is both linearly independent **and** spans V. Via a procedure called **sifting**[1], any collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ that spans V can be sifted to obtain[2] a linearly independent set that still spans V. This sifted subsequence in fact forms a basis for V.

The following notational conventions will be important:

- An ***ordered* basis** for V will be written as an ordered list i.e. a tuple $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.
- Any unordered collection (possibly including repeats) of vectors will be written as $\mathbf{v}_1, \ldots, \mathbf{v}_n$.
- I will try my best to write a basis of vectors as a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to emphasise the lack of repetitions (and lack of ordering).

Proposition 1.1.3 The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ form a basis for V if and only there exists a unique sequence of scalars $\alpha_1, \ldots, \alpha_n$ s.t.
$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n.$$

Definition 1.1.4 The scalars in the above proposition are called the **coordinates of v with respect to the basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of V**.

An important theorem called **The Basis Theorem** asserts that all finite bases of a single vector space V contain the same number of vectors. Consequently, we may unambiguously define the **dimension of V** as the number of vectors in any finite basis of V.

[1] The procedure of **sifting** goes as follows. Write out the collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$. For each $j = 1, \ldots, n$:
- If either $\mathbf{v}_j = \mathbf{0}_V$, or $\mathbf{v}_j$ is a linear combination of its precedent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$, then remove $\mathbf{v}_j$ from the list. Else, leave $\mathbf{v}_j$ as is, consider the next element in the sequence and repeat.

[2] The dual result says that any linearly independent subset of V can be extended to (i.e. is contained in) a basis for V.

1.2 Linear Maps

Let U and V be vector spaces over the same field $\mathbb{K}$. A **linear transformation**, or **linear map**, **from U to V** is a function $T\colon U \to V$ such that for all $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\alpha, \beta \in \mathbb{K}$ the following condition holds

$$T(\alpha\mathbf{u}_1 + \beta\mathbf{u}_2) = \alpha T(\mathbf{u}_1) + \beta T(\mathbf{u}_2).$$

Definition 1.2.1 For fixed vector spaces U and V over $\mathbb{K}$, we denote **the set of all linear maps from U to V by $\mathrm{Hom}_{\mathbb{K}}(U;V)$**. Define the following two binary operations on $\mathrm{Hom}_{\mathbb{K}}(U;V)$

- Addition: Let $T_1, T_2 \in \mathrm{Hom}_{\mathbb{K}}(U;V)$. Define $T_1 + T_2 \colon U \to V$ for all $\mathbf{u} \in U$ by

$$(T_1 + T_2)(\mathbf{u}) := T_1(\mathbf{u}) + T_2(\mathbf{u}).$$

- Scalar multiplication: Let $\alpha \in \mathbb{K}$. Define $\alpha T_1 \colon U \to V$ for all $\mathbf{u} \in U$ by

$$(\alpha T_1)(\mathbf{u}) := \alpha T_1(\mathbf{u}).$$

These two binary operations make $\mathrm{Hom}_{\mathbb{K}}(U;V)$ a vector space over $\mathbb{K}$.

Another approach to define dimension unambiguously is as an invariant of vector spaces preserved by a particular type of linear map between vector spaces:

Definition 1.2.2

- A linear map $T \in \mathrm{Hom}_{\mathbb{K}}(U;V)$ is called a **linear isomorphism** if there exists some $S \in \mathrm{Hom}_{\mathbb{K}}(V;U)$ such that $T \circ S = \mathrm{id}_V$ and $S \circ T = \mathrm{id}_V$.
- If there exists a linear isomorphism between vector spaces $U \to V$, then we say that U and V are isomorphic, and denote this by $U \cong V$.

If a linear map $T \in \mathrm{Hom}_{\mathbb{K}}(U;V)$ is bijective, there's no guarantee a priori that its inverse (which exists) will also be linear. However, we prove that this is indeed the case below (and so a linear isomorphism between vector spaces can be equivalently called a bijective linear map):

Proposition 1.2.3 Let U and V be vector spaces over $\mathbb{K}$, and $T\colon U \to V$. Then T is a linear isomorphism iff T is a bijective linear map.

Proof.

$\Longrightarrow$ By definition.

$\Longleftarrow$ The reverse direction is the non-trivial bit discussed just above the proof. Suppose that $T\colon U \to V$ is a linear bijection. As a function, it has an inverse $S\colon V \to U$ and we wish to show that this is linear to conclude that T is a linear isomorphism: Let $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $\alpha, \beta \in \mathbb{K}$. Then

$$\begin{aligned} T(S(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2)) = (T \circ S)(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) &= \mathrm{id}_V(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) \\ &= \alpha\,\mathrm{id}_V(\mathbf{v}_1) + \beta(\mathbf{v}_2) \quad \text{by linearity of } \mathrm{id}_V \\ &= \alpha(T \circ S)(\mathbf{v}_1) + \beta(T \circ S)(\mathbf{v}_2) \\ &= T(\alpha S(\mathbf{v}_1) + \beta S(\mathbf{v}_s)) \quad \text{since } T \text{ is linear} \end{aligned}$$

Since T is bijective, it's injective, and so $\alpha S(\mathbf{v}_1) + \beta S(\mathbf{v}_2) = S(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2)$ and so S is linear.

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Some useful facts will now be presented and find immediate application when discussing an interpretation of ordered lists of vectors (and hence ordered bases) I was introduced to by `pseudonium` **(whose blog can be found here)**.

Proposition 1.2.4 Let $T \in \mathrm{Hom}_{\mathbb{K}}(U; V)$. Then:

(i) If $T: U \to V$ is injective, and $E \subseteq U$ is a linearly independent subset of U, then $T(E)$ is a linearly independent subset of V.

(ii) If $T: U \to V$ is surjective, and $E \subseteq U$ spans U, then $T(E)$ spans V.

(iii) If $T: U \to V$ is an isomorphism, and $E \subseteq U$ is a basis for U, then $T(E)$ is a basis for V.

Proof.

(i) This will be a proof by contradiction.

Proving $P \implies Q$ by contradiction means that one assumes the implication is false. This forces us into the second row of the following truth table:

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Thus, a proof by contradiction assumes the proposition $P \wedge (\neg Q)$.

Our statement is of the form $(A \wedge B) \implies C$ so a proof by contradiction assumes $(A \wedge B) \wedge (\neg C)$, where:

- A: "T is injective"
- B: "E is linearly independent"
- C: "$T(E)$ is linearly independent"

Proof of (i) Suppose that T is injective, E is a linearly independent subset of U, and $T(E)$ is a linearly *dependent* subset of V. Since $T(E)$ is a linearly dependent subset of V, there exist $\mathbf{e}_1, \ldots, \mathbf{e}_r \in E$, and scalars $\alpha_1, \ldots, \alpha_r \in \mathbb{K}$ not all zero s.t.

$$\sum_{i=1}^{r} \alpha_i T(\mathbf{e}_i) = \mathbf{0}_V$$

Now note that by the linearity of T, we can say a bit more:

$$T\left(\sum_{i=1}^{r} \alpha_i \mathbf{e}_i\right) = \sum_{i=1}^{r} \alpha_i T(\mathbf{e}_i) = \mathbf{0}_V = T(\mathbf{0}_U).$$

Since T is injective, this implies that

$$\sum_{i=1}^{r} \alpha_i \mathbf{e}_i = \mathbf{0}_U$$

which means that E is a linearly dependent subset of U, contradicting the assumption that E is a linearly *independent* subset of U.

(ii) Let $T: U \to V$ be surjective, and denote the collection $E \subseteq U$ by $\mathbf{e}_1, \ldots, \mathbf{e}_r$. Suppose that E spans U. We wish to show that $T(E)$ spans V. Let $\mathbf{v} \in V$. By surjectivity, there exists some $\mathbf{u} \in U$ s.t. $T(\mathbf{u}) = \mathbf{v}$. Since E spans U, there exist some scalars $\alpha_1, \ldots, \alpha_r \in \mathbb{K}$ s.t. $u = \alpha_1 \mathbf{e}_1 + \ldots + \alpha_r \mathbf{e}_r$. By the linearity of T, we conclude that for every $\mathbf{v} \in V$:

$$\mathbf{v} = T(\mathbf{u}) = T\left(\sum_{i=1}^{r} \alpha_i \mathbf{e}_i\right) = \sum_{i=1}^{r} \alpha_i T(\mathbf{e}_i)$$

i.e. $T(E)$ spans V.

(iii) Follows from (i) and (ii). ∎

It follows immediately that dimension is a quantity of finite vector spaces that's invariant under isomorphism:

Corollary 1.2.5 Let U and V be finite-dimensional vector spaces over $\mathbb{K}$. If $U \cong V$, then $\dim(U) = \dim(V)$.

1.2.1 New interpretation of a list of vectors

Start with a simple example. An *ordered* list $(\mathbf{v}_1, \mathbf{v}_2)$ of two vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$ can be thought of as a function from a two-element set to V; simply send the first element to $\mathbf{v}_1$, and the second to $\mathbf{v}_2$. We can extend this function "by linearity" to get a linear map $L\colon \mathbb{K}^{2,1} \to V$ by defining

$$L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a\mathbf{v}_1 + b\mathbf{v}_2.$$

You can go the other way too: if you have a linear map $L\colon \mathbb{K}^{2,1} \to V$, applying it to the ordered standard basis $\mathfrak{e}_1, \mathfrak{e}_2$ of $\mathbb{K}^{2,1}$ gives a list of two vectors in V. Namely,

$$\mathbf{v}_1 := L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \quad \text{and} \quad \mathbf{v}_2 := L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

This is a natural way of converting between functions $\{1, 2\} \to V$ and linear maps $\mathbb{K}^{2,1} \to V$. This discussion remains valid for any ordered list of n vectors. We make this discussion formal by stating it as the following:

Proposition 1.2.6 Suppose that V is a vector space over $\mathbb{K}$ of finite dimension $\dim(V) = n$. Let $(\mathfrak{e}_1, \dots, \mathfrak{e}_n)$ denote the ordered standard basis of $\mathbb{K}^{n,1}$. There is a linear bijection

$$\begin{aligned} \Phi\colon &\{L\colon \mathbb{K}^{n,1} \to V \text{ s.t. } L - \text{isomorphism}\} \to \{(\mathbf{v}_1, \dots, \mathbf{v}_n) \text{ an (ordered) basis of } V\} \\ &: L \longmapsto (L(\mathfrak{e}_1), \dots, L(\mathfrak{e}_n)). \end{aligned}$$

Proof. This is certainly a function because L being a linear isomorphism means it sends a basis to a basis. In our case, the standard basis of $\mathbb{K}^{n,1}$ is sent to a basis of V by L. For injectivity, suppose that L_1 and L_2 are isomorphisms $\mathbb{K}^{n,1} \to V$ s.t. $\Phi(L_1) = \Phi(L_2)$ i.e. $L_1(\mathfrak{e}_i) = L_2(\mathfrak{e}_i)$. We wish to show that $L_1 = L_2$. Observe that

$$L_1\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = L_1\left(\sum_{i=1}^n x_i \mathfrak{e}_i\right) = \sum_{i=1}^n x_i L_1(\mathfrak{e}_i) = \sum_{i=1}^n x_i L_2(\mathfrak{e}_i) = L_2\left(\sum_{i=1}^n x_i \mathfrak{e}_i\right) = L_2\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right).$$

For surjectivity, suppose that $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ is an ordered basis for V. Then define[3]

$$L\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \sum_{i=1}^n x_i \mathbf{v}_i.$$

This is well-defined, and it can be easily shown that it's linear. We wish to show that it's an isomorphism. Suppose that

$$L\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \mathbf{0}_V \quad \text{i.e.} \quad \sum_{i=1}^n x_i \mathbf{v}_i = \mathbf{0}_V.$$

[3]This is just the n-dimensional version of the construction $L(\begin{bmatrix} a & b \end{bmatrix}^\top) = a\mathbf{v}_1 + b\mathbf{v}_2$ in the motivating paragraph at the start of this section.

By the linear independence of the ordered basis, this forces $x_i = 0$ for all i. This means that $\ker(L) = \{\mathbf{0}_{\mathbb{K}^{n,1}}\}$ which is equivalent to L being injective. Notice that

$$\operatorname{image}(L) = \left\{ L\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) \in V : \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{K}^{n,1} \right\} = \left\{ \sum_{i=1}^{n} x_i \mathbf{v}_i : \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{K}^{n,1} \right\} = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n).$$

Since $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ spans V, L is surjective. Thus, L is an isomorphism, and by construction $\Phi(L) = (L(\mathfrak{e}_1), \ldots, L(\mathfrak{e}_n)) = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

■

In summary, one can think of a basis as a linear isomorphism $\mathbb{K}^{n,1} \to V$ that gives us a choice of coordinate system on V.

A wonderful artefact of this this correspondence comes from applying **Proposition 1.2.4** to this correspondence, so that one may describe properties of the list in terms of properties of the corresponding linear map (and vice versa):

- It follows from **Proposition 1.2.4 (ii)** that a list of vectors spans V iff the corresponding linear map is surjective.
- It follows from **Proposition 1.2.4 (i)** that a list of vectors is linearly independent iff the corresponding linear map is injective.
- It follows from **Proposition 1.2.4 (iii)** that the list is a basis iff the corresponding linear map is a linear isomorphism (equiv. a linear bijection between vector spaces)

1.2.2 Defining a linear map on a basis is enough

Linear maps are uniquely determined by their action on a basis.

Proposition 1.2.7 Let U, V be vector spaces over $\mathbb{K}$, let $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ be a basis of U, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be any sequence of n vectors in V. Then, there's a unique linear map $T\colon U \to V$ with $T(\mathbf{u}_i) = \mathbf{v}_i$ for $1 \leqslant i \leqslant n$.

Proof. Let $\mathbf{u} \in U$. By **Proposition 1.1.3**, there exists a unique sequence of scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ s.t. $u = \lambda_1 \mathbf{u}_1 + \ldots + \lambda_n \mathbf{u}_n$. The only possible definition of $T\colon U \to V$ compatible with linearity, if such a map exists at all, must obey

$$T(\mathbf{u}) = T\left(\sum_{i=1}^{n} \alpha_i \mathbf{u}_i\right) := \sum_{u=1}^{n} \alpha_i \mathbf{v}_i$$

for some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$. This definition doesn't depend on any choices of the α_i because they're uniquely determined. Thus, T is well-defined as a function from U to V.

- Is T linear? Let $\mathbf{u}, \mathbf{w} \in U$. Then $u = \alpha_1 \mathbf{u}_1 + \ldots + \alpha_n \mathbf{u}_n$ and $w = \beta_1 \mathbf{u}_1 + \ldots + \beta_n \mathbf{u}_n$ which implies that $\mathbf{u} + \mathbf{w} = \sum_1^n (\alpha_i + \beta_i)\mathbf{u}_i$ and so

$$\begin{aligned} T(\mathbf{u} + \mathbf{w}) &:= \sum_{i=1}^{n} (\alpha_i + \beta_i) \mathbf{v}_i \\ &= \sum_{i=1}^{n} \alpha_i \mathbf{v}_i + \sum_{1}^{n} \beta_i \mathbf{v}_i \\ &=: T(\mathbf{u}) + T(\mathbf{w}). \end{aligned}$$

Also, for any $\alpha \in \mathbb{K}$:

$$T(\alpha \mathbf{u}) := T\left(\sum_{i=1}^{n} \alpha\alpha_i \mathbf{u}_i\right) := \sum_{i=1}^{n} \alpha\alpha_i \mathbf{v}_i = \alpha \sum_{i=1}^{n} \alpha_i \mathbf{v}_i =: \alpha T(\mathbf{u}).$$

- Action on the basis: Every $\mathbf{u}_j$ can be written as $\sum_{i=1}^{n} \delta_{ji}\mathbf{u}_i$ which implies that

$$T(\mathbf{u}_j) := \sum_{i=1}^{n} \delta_{ji}\mathbf{v}_i = \mathbf{v}_j.$$

Therefore, there exists a linear map $T\colon U \to V$ with $T(\mathbf{u}_i) = \mathbf{v}_i$ for all i (for some arbitrary vectors $\mathbf{v}_i$ for $i = 1, \ldots, n$).

- Uniqueness: Suppose that there exists another linear map $S\colon U \to V$ with $S(\mathbf{u}_i) = \mathbf{v}_i$ for $i = 1, \ldots, n$. Let $\mathbf{u} \in U$. Then $\mathbf{u} = \alpha_1\mathbf{u}_1 + \ldots + \alpha_n\mathbf{u}_n$ for some uniquely determined sequence of scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$. By the linearity of S,

$$\begin{aligned} S(\mathbf{u}) := \sum_{i=1}^{n} \alpha_i S(\mathbf{u}_i) &= \sum_{i=1}^{n} \alpha_i \mathbf{v}_i \\ &= \sum_{i=1}^{n} \alpha_i T(\mathbf{u}_i) =: T(\mathbf{u}) \end{aligned}$$

holds for all $\mathbf{u} \in U$. Therefore, $T = S$ as maps.

■

At face-value, this theorem looks important but it's even more than that.

1.3 The Correspondence

Let $T\colon U \to V$ be a linear map, where $\dim(U) = n$ and $\dim(V) = m$. Suppose that we're given a basis $E = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ of U, and a basis $F = (\mathbf{f}_1, \ldots, \mathbf{f}_m)$ of V.

Under the correspondence (**Proposition 1.2.6**) between bases and linear isomorphisms into a finite-dimensional vector space:

- The basis E of U corresponds to a linear isomorphism $\Phi_E\colon \mathbb{K}^{n,1} \to U$ satisfying $\Phi_E(\mathfrak{e}_j) = \mathbf{e}_j$ for each standard basis vector $\mathfrak{e}_j \in \mathbb{K}^{n,1}$ for $j = 1, \ldots, n$.
- The basis F of V corresponds to a linear isomorphism $\Psi_F\colon \mathbb{K}^{m,1} \to V$ satisfying $\Psi_F(\mathfrak{f}_i) = \mathbf{f}_i$ for each standard basis vector $\mathfrak{f}_i \in \mathbb{K}^{m,1}$ for $i = 1, \ldots, m$.

$$\begin{tikzcd} U \arrow[r, "T"] & V \\ \mathbb{K}^{n,1} \arrow[u, "\Phi_E"] \arrow[r, "?"'] & \mathbb{K}^{m,1} \arrow[u, "\Psi_F"'] \end{tikzcd}$$

We can define the map $\widetilde{T}\colon \mathbb{K}^{n,1} \to \mathbb{K}^{m,1}$ by $\widetilde{T} := \Psi_F^{-1} \circ T \circ \Phi_E$. This map $\widetilde{T}$ is certainly linear, and so we remark (by **Proposition 1.2.7**) that $\widetilde{T}$ is completely determined by its action on the standard basis $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ of $\mathbb{K}^{n,1}$ i.e. $\widetilde{T}$ is uniquely determined by the list $(\widetilde{T}(\mathfrak{e}_1), \ldots, \widetilde{T}(\mathfrak{e}_n)) \in (K^{m,1})^n$. Note that for each $j = 1, \ldots, n$:

$$\widetilde{T}(\mathfrak{e}_j) = (\Psi_F^{-1} \circ T \circ \Phi_E)(\mathfrak{e}_j) = \Psi_F^{-1}(T(\mathbf{e}_j)).$$

Since $T(\mathbf{e}_j) \in V$, we can write it uniquely as a linear combination $T(\mathbf{e}_j) = \alpha_{1j}\mathbf{f}_1 + \ldots + \alpha_{mj}\mathbf{f}_m$. Then

$$\widetilde{T}(\mathfrak{e}_j) = \Psi_F^{-1}(T(\mathbf{e}_j)) = \begin{bmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{mj} \end{bmatrix} \in \mathbb{K}^{m,1}$$

which is the column vector of coordinates of $T(\mathbf{e}_j)$ with respect to the basis F of V. Since $\widetilde{T}(\mathfrak{e}_j) \in \mathbb{K}^{m,1}$, we may write it uniquely as

$$\widetilde{T}(\mathfrak{e}_j) = \alpha_{1j}\mathfrak{f}_1 + \ldots + \alpha_{mj}\mathfrak{f}_m,$$

and so for any $\underline{\mathbf{x}} \in \mathbb{K}^{n,1}$ we have, by the linearity of $\widetilde{T}$, that:

$$\begin{aligned}
\widetilde{T}(\underline{\mathbf{x}}) &= \widetilde{T}(x_1\mathfrak{e}_1 + \ldots + x_n\mathfrak{e}_n) \\
&= x_1\widetilde{T}(\mathfrak{e}_1) + \ldots + x_n\widetilde{T}(\mathfrak{e}_n) \\
&= x_1(\alpha_{11}\mathfrak{f}_1 + \ldots + \alpha_{m1}\mathfrak{f}_m) + \ldots + x_n(\alpha_{1n}\mathfrak{f}_1 + \ldots + \alpha_{mn}\mathfrak{f}_m) \\
&= (x_1\alpha_{11} + \ldots + x_n\alpha_{1n})\mathfrak{f}_1 + \ldots + (x_1\alpha_{m1} + \ldots + x_n\alpha_{mn})\mathfrak{f}_m \\
&= \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.
\end{aligned}$$

If we let $A = (\alpha_{ij})$, then we see that

$$\widetilde{T}(\underline{\mathbf{x}}) = A\underline{\mathbf{x}}.$$

This means that $\widetilde{T}$ is the map $\underline{\mathbf{x}} \longmapsto A\underline{\mathbf{x}}$ i.e. A is the matrix of scalars that determines $\widetilde{T}$.

Definition 1.3.1

- These coefficients α_{ij} in the j^{th} column of A are called the coordinates of $T(\mathbf{e}_j)$ with respect to the basis $\mathbf{f}_1, \ldots, \mathbf{f}_m$.

  $$T(\mathbf{e}_j) = \sum_{i=1}^{m} \alpha_{ij}\mathbf{f}_i \quad \text{for } j = 1, \ldots, n.$$

  This matrix A is called the **matrix of the linear map T with respect to the chosen bases E of U and F of V**.

- We will denote A by $[F, T, E]$.

- The map $\widetilde{T}$ is called the **coordinate representation of T with respect to the bases E and V**, and will denote by $[T]_E^F$.

- Let $[E, \mathbf{v}]$ denote the coordinate vector of $\mathbf{v}$ with respect to the ordered basis E. If the basis is clear, then sometimes $\underline{\mathbf{v}}$ will be used.

Theorem 1.3.2 Let U and V be vector spaces over $\mathbb{K}$ of dimensions n and m respectively. Then, for a given choice of bases of U and V, there is a one-to-one correspondence[4] between the set $\mathrm{Hom}_{\mathbb{K}}(U; V)$ of linear maps $U \to V$, and the set $\mathbb{K}^{m,n}$ of $m \times n$ matrices over $\mathbb{K}$.

$$\mathrm{Hom}_{\mathbb{K}}(U; V) \cong \mathbb{K}^{m,n}$$

[4]It turns out that this correspondence is indeed a linear map itself!

Now we write the above in slightly simpler notation, connecting how a linear transformation acts on elements of a vector space to how its matrix acts on said vector's coordinates. For the given basis $E = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ of U and a vector $\mathbf{u} = \lambda_1\mathbf{e}_1 + \ldots + \lambda_n\mathbf{e}_n$, let $\underline{\mathbf{u}}$ or $[E, \mathbf{u}]$ denote the column vector whose entries are the coordinates of $\mathbf{u}$ with respect to E i.e.

$$\underline{\mathbf{u}} = [E, \mathbf{u}] = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \in \mathbb{K}^{n,1}.$$

Analogously, for the given basis $F = (\mathbf{f}_1, \ldots, \mathbf{f}_m)$ of V and a vector $\mathbf{v} = \mu_1\mathbf{f}_1 + \ldots + \mu_m\mathbf{f}_m \in V$, let $\underline{\mathbf{v}}$ denote the column vector

$$\underline{\mathbf{v}} = [F, \mathbf{v}] = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \in \mathbb{K}^{m,1}.$$

Proposition 1.3.3 Let $T\colon U \to V$ be a linear map, E a basis of U, and F a basis of V. Let $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$ be the column vectors of coordinates of $\mathbf{u}$ and $\mathbf{v}$ with respect to the bases E and F respectively. Then $T(\mathbf{u}) = \mathbf{v}$ if and only if $[F, T, E]\underline{\mathbf{u}} = \underline{\mathbf{v}}$. In other words, $\underline{T(\mathbf{u})} = [F, T, E]\underline{\mathbf{u}}$.

This proposition is telling us that choosing a basis E for U gives every vector in U a unique set of coordinates (and similarly for V) so applying the linear transformation T to $\mathbf{u} \in U$ is "the same" as multiplying its column vector coordinates $\underline{\mathbf{u}}$ by $[F, T, E]$ as long as we interpret the resulting column vector as coordinates in V with respect to F.

1.4 The Inverse Image Problem

Historically speaking, linear algebra seems to have come from the study and solution of systems of simultaneous linear equations. Consider a system of m equations in n unknowns $x_1, \ldots, x_n$ where $m, n > 1$

$$\begin{cases} \alpha_{11}x_1 & + & \alpha_{12}x_2 & + & \ldots & + & \alpha_{1n}x_n & = & \beta_1 \\ \alpha_{21}x_1 & + & \alpha_{22}x_2 & + & \ldots & + & \alpha_{2n}x_n & = & \beta_2 \\ & & & & & & & \vdots & \\ \alpha_{m1}x_1 & + & \alpha_{m2}x_2 & + & \ldots & + & \alpha_{mn}x_n & = & \beta_m \end{cases}$$

where all coefficients α_{ij} and β_i belong to $\mathbb{K}$. Solving this system means finding all collections $x_1, x_2, \ldots, x_n \in \mathbb{K}$ such that the equations above hold. Let $A = (\alpha_{ij}) \in \mathbb{K}^{m,n}$ be the $m \times n$ matrix of coefficients. The crucial step is to define the column vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{K}^{n,1} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} \in \mathbb{K}^{m,1}$$

which allows us to write our system as a single matrix equation $A\mathbf{x} = \mathbf{b}$, where the coefficient A is a matrix.

Using the notation of linear maps, we've just reduced solving a system of linear equations to the **inverse image problem**. That is, given a linear map $T\colon U \to V$, and a fixed vector $\mathbf{v} \in V$, find all $\mathbf{u} \in U$ s.t. $T(\mathbf{u}) = \mathbf{v}$.

In fact, these problems are equivalent!

In the opposite direction, suppose that we're given the inverse image problem to solve. Choose bases E and F of U and V respectively. Denote by:

- A the corresponding matrix $[F, T, E]$,

- $\mathbf{b} = [E, \mathbf{u}]$ the vector of coordinates of $\mathbf{u}$ with respect to E, and
- $\mathbf{b} = [F, \mathbf{v}]$ the vector of coordinates of $\mathbf{v}$ with respect to F.

By **Proposition 1.3.3**, $T(\mathbf{u}) = \mathbf{v}$ iff $A\mathbf{x} = \mathbf{b}$. This reduces the inverse image problem to solving a system of linear equations.

I want to make some remarks about the inverse image problem but they require the concepts of the image, kernel, (row = column) rank, nullity of a linear map $T\colon U \to V$ (and the nullspace and column space of its corresponding matrix A). I'll cover the important bits before making some remarks about the inverse image problem. If known, please skip to **Remarks 1.4.3**.

Definition 1.4.1 Let $T\colon U \to V$ be a linear map from a finite-dimensional vector space U.

- The **kernel of T** is the set of all vectors in U sent to $\mathbf{0}_V$ i.e.
$$\ker(T) \coloneqq \{\mathbf{u} \in U : T(\mathbf{u}) = \mathbf{0}_V\}.$$
- The **image of T** is the collection of all vectors that are mapped to by some element of U i.e.
$$\text{image}(T) \coloneqq \{\mathbf{v} \in V : \exists \mathbf{u} \in U \text{ s.t. } T(\mathbf{u}) = \mathbf{v}\}.$$
- The **rank of T** is the dimension of its image.
- The **nullity of T** is the dimension of its kernel.

I cannot understate the importance of the following theorem. I will make use of it repeatedly in later sections.

Theorem 1.4.2 (**Rank-Nullity**) Let U, V be vector space over $\mathbb{K}$ with U finite-dimensional, and let $T\colon U \to V$ be a linear map. Then
$$\dim(U) = \text{rank}(T) = \text{nullity}(T).$$

Proof. Since U is finite-dimensional and $\ker(T)$ is a vector subspace of U, then $\ker(T)$ is also finite-dimensional. Let $\text{nullity}(T) = s$, and let $\mathbf{e}_1, \ldots, \mathbf{e}_s$ be a basis of $\ker(T)$. We can extend this sequence of vectors (which are linearly independent in U as required by the theorem) to a basis[5] $\mathbf{e}_1, \ldots, \mathbf{e}_s, \mathbf{f}_1, \ldots, \mathbf{f}_r$ of U. Then $\dim(U) = s + r$. To prove the theorem, we have to prove that $\dim(\text{image}(T)) = r$. Clearly, $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_s), T(\mathbf{f}_1), \ldots, T(\mathbf{f}_r)$ span $\text{image}(T)$ because every $\mathbf{u} \in U$ can be written uniquely as
$$\mathbf{u} = \sum_{i=1}^{s} \alpha_i \mathbf{e}_i + \sum_{j=1}^{r} \beta_j \mathbf{f}_j$$
which implies that any vector $T(\mathbf{u})$ in $\text{image}(T)$ is of the form
$$T(\mathbf{u}) = \sum_{i=1}^{s} \alpha T(\mathbf{e}_i) + \sum_{j=1}^{r} \beta_j T(\mathbf{f}_j).$$
Since $T(\mathbf{e}_1) = \ldots = T(\mathbf{e}_s) = \mathbf{0}_V$, this implies that $T(\mathbf{f}_1), \ldots, T(\mathbf{f}_r)$ span $\text{image}(T)$. If we can demonstrate that they're linearly independent, that concludes the proof.

Suppose that for some scalars α_i we have that
$$\alpha_1 T(\mathbf{f}_1) + \ldots + \alpha_r T(\mathbf{f}_r) = \mathbf{0}_V.$$
Then $T(\alpha_1 \mathbf{f}_1 + \ldots + \alpha_r \mathbf{f}_r) = \mathbf{0}_V$ by linearity so $\alpha_1 \mathbf{f}_1 + \ldots + \alpha_r \mathbf{f}_r \in \ker(T)$. However, $\mathbf{e}_1, \ldots, \mathbf{e}_s$ is a basis of $\ker(T)$ so there exist scalars β_i with
$$\alpha_1 \mathbf{f}_1 + \ldots + \alpha_r \mathbf{f}_r = \beta_1 \mathbf{e}_1 + \ldots + \beta_s \mathbf{e}_s$$

[5] These vectors $\mathbf{f}_1, \ldots, \mathbf{f}_r$ are the collection of vectors remaining after taking a spanning set $\{\mathbf{f}_j\}$ of U, and then sifting the combined sequence $\mathbf{e}_1, \ldots, \mathbf{e}_s, \mathbf{f}_1, \ldots, \mathbf{f}_j$ (with $j \geqslant r$) to get a basis of U.

$$\text{i.e.} \quad \beta_1 \mathbf{e}_1 + \ldots + \beta_s \mathbf{e}_s + (-\alpha_1)\mathbf{f}_1 + \ldots + (-\alpha_r)\mathbf{f}_r = \mathbf{0}_V.$$

Since $\mathbf{e}_1, \ldots, \mathbf{e}_s, \mathbf{f}_1, \ldots, \mathbf{f}_r$ form a basis for U, they are linearly independent which forces $\beta_1 = \ldots = \beta_s = \alpha_1 = \ldots = \alpha_r$, and so we've shown that $T(\mathbf{f}_1), \ldots, T(\mathbf{f}_r)$ are linearly independent. Finally, $\dim(\text{image}(T)) = r$ and the claim is proven. ■

Remarks 1.4.3

- The case when $\mathbf{v} = \mathbf{0}_V$, or equivalently (upon choosing bases E for U, and F for V respectively[6]) when $\beta_i = 0$ for $1 \leqslant i \leqslant m$, is called the **homogeneous case**.

  The set of solutions $\{\mathbf{u} \in U : T(\mathbf{u}) = \mathbf{0}_V\}$ is precisely the kernel $\ker(T)$ of T. The corresponding set of column vectors $\mathbf{x} \in \mathbb{K}^{n,1}$ with $A\mathbf{x} = \mathbf{0}_V$ is the image of $\ker(T)$ under the correspondence. We call this collection the **nullspace** of the matrix A. Since the correspondence is a bijection (i.e. choosing a basis E for U uniquely defines a coordinate vector $\mathbf{x} = [E, \mathbf{u}]$ of $\mathbf{u}$ with respect to E), the nullspace of A is a vector subspace of $\mathbb{K}^{n,1}$ with the same dimension as $\ker(T)$ (which is a subspace of U). Thus, we may define the **nullity of A** as the dimension of its nullspace.

- For the general case $A\mathbf{x} = \mathbf{b}$, it's easy to see that if $\mathbf{x}$ is one solution to a system of linear equations, then the complete set of solutions is equal to

$$\mathbf{x} + \text{nullspace}(A) := \{\mathbf{x} + \mathbf{y} : \mathbf{y} \in \text{nullspace}(A)\}.$$

  Some remarks about the nature of solutions:

  - It's possible that there are no solutions at all: when $\mathbf{v} \notin \text{image}(T)$.
  - If there are solutions, then there is a unique solution precisely when $\ker(T) = \{\mathbf{0}_U\}$, or equivalently when $\text{nullspace}(A) = \{0_{\mathbb{K}^{n,1}}\}$.
  - If $\mathbb{K}$ is an infinite field (like $\mathbb{R}$ or $\mathbb{C}$), and there are solutions but $\ker(T) \neq \{\mathbf{0}_U\}$, then there are infinitely many solutions. (Let $\tilde{\mathbf{x}}$ be a solution of $A\mathbf{x} = \mathbf{b}$. If $\mathbf{y} \in \text{nullspace}(A)$, then $\alpha\mathbf{y} \in \text{nullspace}(A)$ for any $\alpha \in \mathbb{K}$, and so $\tilde{\mathbf{x}} + \alpha\mathbf{y}$ is also a solution.)

1.5 Change of Basis

The matrix representation of a linear map $T : U \to V$ between vector spaces depends on the choice of bases for U and V. (Assume throughout that all vector spaces are over the same field $\mathbb{K}$)

1.5.1 Two bases of the same space

Let U be a vector space of dimension n, and let $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $E' = \{\mathbf{e}'_1, \ldots, \mathbf{e}'_n\}$ be two bases of U. We can convert the coordinate representation of the abstract vector $\mathbf{u}$ with respect to the "old" basis E to the coordinates of $\mathbf{u}$ with respect to the "new" basis E'. Simply take the discussion in **Section 1.3** and adapt it as the following diagram suggests with $T = \text{id}_U$:

$$\begin{tikzcd}
U \arrow[r, "\text{id}_U"] & U \\
\mathbb{K}^{n,1} \arrow[u, "\Phi_E"] \arrow[r, "?"'] & \mathbb{K}^{n,1} \arrow[u, "\Phi_{E'}"']
\end{tikzcd}$$

[6] In fact, it doesn't really matter which basis F for V we choose *in the homogeneous case* — the zero vector $\mathbf{0}_V$ has the same coordinates (all zeroes) with respect to any basis of V.

We also give a name to the map at the bottom of the diagram. Define the **change-of-basis map $\left[\mathrm{id}_U\right]_E^{E'} : \mathbb{K}^{n,1} \to \mathbb{K}^{n,1}$ from E to E'** by

$$\left[\mathrm{id}_U\right]_E^{E'} := (\Phi_{E'}^{-1} \circ \mathrm{id}_U \circ \Phi_E) = \Phi_{E'}^{-1} \circ \Phi_E.$$

For each $\mathfrak{e}_j \in \mathbb{K}^{n,1}$, we have that

$$\left[\mathrm{id}_U\right]_E^{E'}(\mathfrak{e}_j) = \Phi_{E'}^{-1}(\Phi_E(\mathfrak{e}_j)) = \Phi_{E'}^{-1}(\mathbf{e}_j)$$

which is the coordinate vector of $\mathbf{e}_i$ with respect to the basis E'. As before, the matrix representation of $\left[\mathrm{id}_U\right]_E^{E'}$ is denoted by $[E', \mathrm{id}_U, E] = P = (\sigma_{ij})$ where the j^{th} column of P is $\Phi_{E'}^{-1}(\mathbf{e}_j)$.

Definition 1.5.1 Let U be a vector space of dimension n, and let $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $E' = \{\mathbf{e}'_1, \ldots, \mathbf{e}'_n\}$ be two bases of U. The matrix $P = [E', \mathrm{id}_U, E]$ of the identity map $\mathrm{id}_U : U \to U$ with respect to the bases E in the domain, and E' in the codomain respectively, is called the **change-of-basis matrix from the basis E to the basis E'**.

Remarks 1.5.2

- The name *change-of-basis matrix* is justified by the fact that if $\mathbf{u} \in U$, and we let $\underline{\mathbf{u}}$ and $\underline{\mathbf{u}'}$ denote the column vectors of coordinates with respect to the bases E and E' respectively, then $\underline{\mathbf{v}'} = P\underline{\mathbf{v}}$. It's a matrix that turns a vector $\mathbf{u}$'s coordinates with respect to the old basis into the same vector $\mathbf{u}$'s coordinates with respect to the new basis.
- Every change-of-basis matrix is invertible.
- Conversely, every invertible matrix *can* be viewed[7] as a change-of-basis matrix upon the choice of suitable bases i.e. if $P \in \mathbb{K}^{n,n}$ is invertible, then there exist bases E and E' of $\mathbb{K}^{n,1}$ s.t. P is the change-of-basis matrix from E to E'.

1.5.2 The effect of change-of-basis on linear maps $T : U \to V$

Let U be a vector space with $\dim(U) = n < \infty$, V be a vector space with $\dim(V) = m < \infty$, and let $T \in \mathrm{Hom}_{\mathbb{K}}(U; V)$.

- Let $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for U.
  - E corresponds to a linear isomorphism $\Phi_E : \mathbb{K}^{n,1} \to U$ satisfying $\Phi_E(\mathfrak{e}_j) = \mathbf{e}_j$ for each standard basis vector $\mathfrak{e}_j \in \mathbb{K}^{n,1}$ for $j = 1, \ldots, n$
- Let $F = \{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$ be a basis for V.
  - F corresponds to a linear isomorphism $\Psi_F : \mathbb{K}^{m,1} \to U$ satisfying $\Psi_F(\mathfrak{f}_j) = \mathbf{f}_i$ for each standard basis vector $\mathfrak{f}_i \in \mathbb{K}^{m,1}$ for $i = 1, \ldots, m$

Then

$$T(\mathbf{e}_j) = \sum_{i=1}^{m} \alpha_{ij}\mathbf{f}_i \quad \text{for } 1 \leqslant j \leqslant n$$

where $A = (\alpha_{ij}) = [F, T, E]$ is the $m \times n$ matrix of T with respect to the bases E of U, F of V.

Now choose new bases.

- Let $E' = \{\mathbf{e}'_1, \ldots, \mathbf{e}'_n\}$ be a "new" basis of U.
  - E' corresponds to a linear isomorphism $\Phi_{E'} : \mathbb{K}^{n,1} \to U$ satisfying $\Phi_{E'}(\mathfrak{e}'_j) = \mathbf{e}'_j$ for each standard basis vector $\mathfrak{e}'_j \in \mathbb{K}^{n,1}$ for $j = 1, \ldots, n$

[7] In general, an invertible matrix doesn't have to be viewed as a change-of-basis matrix. This is a subtle but important distinction to keep the reader away from the trap (that I briefly fell into) of thinking that every invertible matrix **is** a change-of-basis matrix without explicit reference to a choice of bases.

- Let $F' = \{\mathbf{f}'_1, \ldots, \mathbf{f}'_m\}$ be a "new" basis of V.
    - F' corresponds to a linear isomorphism $\Psi_{F'} : \mathbb{K}^{m,1} \to U$ satisfying $\Psi_{F'}(\mathfrak{f}'_i) = \mathbf{f}'_i$ for each standard basis vector $\mathfrak{f}'_i \in \mathbb{K}^{m,1}$ for $i = 1, \ldots, m$

There is now a matrix representing the transformation T

$$T(\mathbf{e}'_j) = \sum_{i=1}^{m} \beta_{ij} \mathbf{f}'_i \quad \text{for } 1 \leqslant j \leqslant n$$

where $B = (\beta_{ij}) = [F', T, E']$ is the $m \times n$ matrix of T with respect to the bases E' of U, F' of V.

Our objective is to find the relationship between the matrices A and B in terms of the change-of-basis matrices. We draw the following diagram:

$$\begin{tikzcd}
\mathbb{K}^{n,1} \arrow[rrr, "{\left[T\right]_{E'}^{F'} = (\underline{\mathbf{x}} \longmapsto B\underline{\mathbf{x}})}"] \arrow[dr, "\Phi_{E'}"] & & & \mathbb{K}^{m,1} \arrow[dl, "\Psi_{F'}"'] \\
& U \arrow[r, "T"] & V & \\
& U \arrow[u, "\mathrm{id}_U"] \arrow[r, "T"'] & V \arrow[u, "\mathrm{id}_V"'] & \\
\mathbb{K}^{n,1} \arrow[uuu, "{\left[\mathrm{id}_U\right]_{E}^{E'}}"] \arrow[ur, "\Phi_E"'] \arrow[rrr, "{\left[T\right]_{E}^{F} = (\underline{\mathbf{x}} \longmapsto A\underline{\mathbf{x}})}"'] & & & \mathbb{K}^{m,1} \arrow[uuu, "{\left[\mathrm{id}_V\right]_{F}^{F'}}"'] \arrow[ul, "\Psi_F"]
\end{tikzcd}$$

Let P and Q denote the matrices of the change-of-basis maps $\left[\mathrm{id}_U\right]_E^{E'}$ from E to E', and $\left[\mathrm{id}_V\right]_F^{F'}$ from F to F', respectively. It follows that:

$$\left[T\right]_{E'}^{F'} = \left[\mathrm{id}_V\right]_F^{F'} \circ \left[T\right]_E^F \circ \left(\left[\mathrm{id}_U\right]_E^{E'}\right)^{-1}$$

which in the language of matrices (where composition corresponds to multiplication of matrices) is

$$B = QAP^{-1}.$$

Theorem 1.5.3 With the above notation, we have $QA = BP$, or equivalently $B = QAP^{-1}$.

Remarks 1.5.4 The matrix B is the matrix which, given the coordinates of a vector $\mathbf{u} \in U$ with respect to the basis E' produces the coordinates of $T(\mathbf{u}) \in V$ with respect to the basis F'.

On the other hand, suppose we already know the matrix A which performs the corresponding task with the "old" bases E and F. Now, given the coordinates of some vector $\mathbf{u}$ with respect to the "new" basis, we need to:

- Find the coordinates of $\mathbf{u}$ with respect to the "old" basis of U: this is done by multiplying by the change-of-basis matrix from E' to E, which is P^{-1};
- find the coordinates of $T(\mathbf{u})$ with respect to the "old" basis of V: this is what multiplying by A does;

- translate the result into coordinates with respect to the "new" basis for V: this is done by multiplying by the change-of-basis matrix Q.

Putting these three steps together

$$B[E', \mathbf{u}] = QAP^{-1}[E', \mathbf{u}].$$

Corollary 1.5.5 Two $m \times n$ matrices A and B represent the same linear map from an n-dimensional space U to an m-dimensional space V (with respect to different bases) if and only if there exist invertible matrices P (of size $n \times n$) and Q (of size $m \times m$) with $B = QAP^{-1}$.

Definition 1.5.6 Two matrices **A and B of size $m \times n$** are said to be **equivalent** if there exist invertible P and Q with $B = QAP^{-1}$.

Other books will use $B = QAP$ in their general definition but I think this obfuscates the original nature of change-of-basis matrices.

Theorem 1.5.7 Equivalence of $m \times n$ matrices defines an equivalence relation on $\mathbb{K}^{m,n}$.

This begs the question, what is a good choice of representative for an equivalence class of equivalent matrices? The following theorem gives a canonical form that serves as an easily identifiable representative of any such equivalence class.

Theorem 1.5.8 Let A and B be $m \times n$ matrices over $\mathbb{K}$. Then the following conditions on A and B are equivalent:

(i) A and B are equivalent.

(ii) A and B represent the same linear map with respect to different bases.

(iii) A and B have the same rank.

(iv) B can be obtained from A by application of elementary row and column operations.

The proof of this theorem requires the fact that any $m \times n$ matrix A can be brought, by elementary row[8] and elementary column operations, into the **Smith normal form E_s**

$$E_s = \left[\begin{array}{c|c} I_s & 0_{s,\,n-s} \\ \hline 0_{m-s,\,s} & 0_{m-s,\,n-s} \end{array}\right].$$

Another key observation required to complete the proof is that an elementary row operation corresponds to left-multiplication by an elementary row matrix. An analogous statement applies for column operations but replace 'left' with 'right'-multiplication.

Accordingly, in the proof that (iv) $\implies$ (i), suppose that B can be obtained from A by row operations represented by $R_1, \ldots, R_r$ and elementary column operations represented by $C_1, \ldots, C_t$. This means that $B = R_r \ldots R_1 A C_1 \ldots C_t$. Elementary matrices are invertible so if we let $Q = R_r \ldots R_1$ and $P^{-1} = C_1 \ldots C_t$, then $B = QAP^{-1}$.

Proposition 1.5.9 Any $m \times n$ matrix is equivalent to the matrix E_s defined above, where $s = \operatorname{rank}(A)$.

[8] Let A be an $m \times n$ matrix over $\mathbb{K}$ with rows $\mathbf{r}_1, \ldots, \mathbf{r}_m \in \mathbb{K}^{1,n}$. The three types of **elementary row operation on A** are defined as follows:

(R1) For some $i \neq j$, add a multiply of $\mathbf{r}_j$ to $\mathbf{r}_i$.

(R2) Interchange two rows.

(R3) Multiply a row by a non-zero scalar.

The definition for **elementary column operations** is entirely analogous — replace 'row' with 'column'.

1.6 Similar Matrices

Now we look at the case of what happens to the matrix of a **linear operator** (a linear map from V to itself) $T\colon V \to V$ when applying a single change-of-basis.

Let V be a vector space of dimension n over the field $\mathbb{K}$, and $T\colon V \to V$ be a linear operator. Let $E = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ and $E' = (\mathbf{e}'_1, \ldots, \mathbf{e}'_n)$ be bases of V, and $A = (\alpha_{ij}) = [E, T, E]$ and $B = (\beta_{ij}) = [E', T, E']$ be the matrices of T with respect E and E' respectively.

By the discussion leading up to **Theorem 1.5.3**, the following diagram describes the current setup:

$$\begin{tikzcd}
\mathbb{K}^{n,1} \arrow[rrr, "{\left[T\right]^{E'}_{E'} = (\underline{\mathbf{x}} \longmapsto B\underline{\mathbf{x}})}"] \arrow[dr, "\Phi_{E'}"] & & & \mathbb{K}^{n,1} \arrow[dl, "\Phi_{E'}"'] \\
& V \arrow[r, "T"] & V & \\
& V \arrow[u, "\mathrm{id}_V"] \arrow[r, "T"'] & V \arrow[u, "\mathrm{id}_V"'] & \\
\mathbb{K}^{n,1} \arrow[uuu, "{\left[\mathrm{id}_V\right]^{E'}_{E}}"] \arrow[ur, "\Phi_E"] \arrow[rrr, "{\left[T\right]^{E}_{E} = (\underline{\mathbf{x}} \longmapsto A\underline{\mathbf{x}})}"'] & & & \mathbb{K}^{n,1} \arrow[uuu, "{\left[\mathrm{id}_V\right]^{E'}_{E}}"'] \arrow[ul, "\Phi_E"']
\end{tikzcd}$$

where P denotes the matrix of the change-of-basis map $\left[\mathrm{id}_V\right]^{E'}_{E}$ from E to E'. It follows that:

$$\left[T\right]^{E'}_{E'} = \left[\mathrm{id}_V\right]^{E'}_{E} \circ \left[T\right]^{E}_{E} \circ \left(\left[\mathrm{id}_V\right]^{E'}_{E}\right)^{-1}$$

which in the language of matrices corresponds to

$$B = PAP^{-1}.$$

Definition 1.6.1 Two $n \times n$ matrices A and B over $\mathbb{K}$ are said to be **similar** if there exists an invertible $n \times n$ matrix P s.t. $B = PAP^{-1}$.

Thus, two matrices are similar if and only if they represent the same linear operator $T\colon V \to V$ with respect to different bases of V.

Similarity is an equivalence relation on the set of $n \times n$ matrices over $\mathbb{K}$. Apparently it's not as straight-forward to pick a canonical representative from an equivalence class because there is some dependence on the field $\mathbb{K}$. For $\mathbb{K} = \mathbb{C}$, we have the "Jordan Canonical Form".

We could equivalently have cut down on some of the bloat in the diagram and drawn something more succinct:

$$\begin{tikzcd}
\mathbb{K}^{n,1} \arrow[rrr, "{\left[T\right]^{E'}_{E'} = (\underline{\mathbf{x}} \longmapsto B\underline{\mathbf{x}})}"] \arrow[dr, "\Phi_{E'}"] & & & \mathbb{K}^{n,1} \arrow[dl, "\Phi_{E'}"'] \\
& V \arrow[r, "T"] & V & \\
\mathbb{K}^{n,1} \arrow[uu, "{\left[\mathrm{id}_V\right]^{E'}_{E}}"] \arrow[ur, "\Phi_E"] \arrow[rrr, "{\left[T\right]^{E}_{E} = (\underline{\mathbf{x}} \longmapsto A\underline{\mathbf{x}})}"'] & & & \mathbb{K}^{n,1} \arrow[uu, "{\left[\mathrm{id}_V\right]^{E'}_{E}}"'] \arrow[ul, "\Phi_E"']
\end{tikzcd}$$

1.7 Diagonalisation

Definition 1.7.1

- A square matrix $D = (d_{ij}) \in \mathbb{K}^{n,n}$ is called **diagonal** if all its entries off the main diagonal are zero i.e. $d_{ij} = 0$ for $i \neq j$.
- A matrix which is similar to a diagonal matrix is said to be **diagonalisable**.
- Let $T\colon V \to V$ be a linear operator. We say that **T is diagonalisable** if there exists an ordered basis E of V with respect to which $[E, T, E]$ is a diagonal matrix.

I'm not entirely sure on the historical motivation for investigating whether a linear operator $T\colon V \to V$ on a finite dimensional vector space V over K is diagonalisable or not, but I imagine it has something do with simplifying some physical system to see which bits only scale with application of T. The reason for this hunch is as follows:

Note that if $[E, T, E] = D$ is a diagonal matrix, then for each vector $\mathbf{e}_j \in E$ we observe that

$$T(\mathbf{e}_j) = \sum_{i=1}^{n} d_{ij}\mathbf{e}_i = d_{jj}\mathbf{e}_j.$$

Conversely, if $E = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is an ordered basis for V s.t. $T(\mathbf{e}_j) = \lambda_j \mathbf{e}_j$ for some scalars $\lambda_1, \ldots, \lambda_n$, then clearly

$$[E, T, E] = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Each vector in the above basis satisfies $T(\mathbf{v}) = \lambda\mathbf{v}$ for some scalar λ. Since $\mathbf{v}$ is a basis element, it's non-zero. This seemingly motivates the following definitions (of eigenvalues and eigenvectors).

It turns out that the possible entries on the diagonal of a matrix similar to A can be calculated directly from A — they are called *eigenvalues of A* and *depend only on the linear map to which A corresponds*, and *not* on the particular choice of basis.

- Let $T\colon V \to V$ be a linear map, where V is a vector space over $\mathbb{K}$. Suppose that for some non-zero vector $v \in V$ and some scalar $\lambda \in \mathbb{K}$, we have $T(v) = \lambda v$. Then v is called an **eigenvector of T**, and λ is called the **eigenvalue of T corresponding to v**.
    - Equivalently, by **Proposition 1.3.3** we know that $T(\mathbf{u}) = \mathbf{v} \iff \underline{T(\mathbf{u})} = [F, T, E]\underline{\mathbf{u}}$. In particular, $T(\mathbf{v}) = \lambda\mathbf{v} \iff A\underline{\mathbf{v}} = \lambda\underline{\mathbf{v}}$.

    Definition 1.7.2 Let A be an $n \times n$ matrix over $\mathbb{K}$. Suppose that for some non-zero column vector $\underline{\mathbf{v}} \in \mathbb{K}^{n,1}$, and some scalar $\lambda \in \mathbb{K}$ we have $A\underline{\mathbf{v}} = \lambda\underline{\mathbf{v}}$. Then λ is called an eigenvalue of A, and $\underline{\mathbf{v}}$ is called the eigenvector of A corresponding to λ.

    Thus, if A is the matrix that corresponds to T under a choice of basis of V, then λ is an eigenvalue of T iff λ is an eigenvalue of A.

- Geometrically speaking, eigenvectors of a linear operator $T\colon V \to V$ are vectors in V that are merely elongated or shrunk under T (with scale factor the corresponding eigenvalue λ).
- Another important remark is that because similar matrices represent the same linear map with respect to different bases, they have the same eigenvalues. We will prove this after figuring out the following question:
- How do we calculate eigenvalues?

Theorem 1.7.3 Let A be an $n \times n$ matrix. Then λ is an eigenvalue of A iff $\det(A - \lambda I_n) = 0$.

Treating λ as unknown, we get a polynomial equation which we can solve to find all the eigenvalues of A:

Definition 1.7.4 For an $n \times n$ matrix A, the equation

$$\det(A - xI_n) = 0$$

is called the **characteristic equation of A**, and $c_A(x) := \det(A - xI_n)$ is called **a characteristic polynomial of A** — a polynomial in x of degree n with leading coefficient $(-1)^n$.

- **Similar matrices have the same characteristic equation**, and hence the same eigenvalues!

Proof. Let A and B be similar matrices i.e. there exists an invertible matrix P s.t. $B = PAP^{-1}$. Then

$$\begin{aligned}\det(B - xI_n) &= \det(PAP^{-1} - xI_n)\\ &= \det(P(A - xI_n)P^{-1})\\ &= \det(P)\det(A - xI_n)\det(P^{-1})\\ &= \det(A - xI_n).\end{aligned}$$

■

Since similar matrices correspond to the same linear operator on V with respect to different bases, similar matrices have the same characteristic polynomial. Thus, we may unambiguously define the characteristic polynomial of T to be the characteristic polynomial of a representative A of the equivalence class of matrices similar to A i.e.

Definition 1.7.5 Let V be an n-dimensional vector space V over $\mathbb{K}$, and choose a basis E of V. We define **the characteristic polynomial of a linear operator $T\colon V \to V$**, denoted by $c_T(x)$ or "$\det(T - x\,\mathrm{id}_V)$", to be the characteristic polynomial of $A = [E, T, E]$ i.e.

$$f(x) := c_A(x) := \det(A - xI_n).$$

- There is one case where the eigenvalues can be written down immediately:

Definition 1.7.6 A square matrix $A \in \mathbb{K}^{n,n}$ is called **upper triangular** if all of its entries below the main diagonal are zero i.e. $\alpha_{ij} = 0$ for $i > j$.

Proposition 1.7.7 Suppose that the matrix A is upper triangular. Then the eigenvalues of A are just the diagonal entries α_{ii} of A.

The proof relies on some intermediate facts:

Theorem 1.7.8 (The effect of matrix operations on the determinant)

(i) $\det(I_n) = 1$

(ii) Let B result from A by applying (R2) i.e. interchanging two rows. Then $\det(B) = -\det(A)$.

(iii) If A has two equal rows, then $\det(A) = 0$.

(iv) Let B result from A by applying (R1) i.e. adding a multiple of one row to another. Then $\det(B) = \det(A)$.

(v) Let B result from A by applying (R3) i.e. multiplying a row by a scalar λ. Then $\det(B) = \lambda\det(A)$.

All of the above theorem holds if we replace rows by columns because it can be shown (from the definition of the determinant) that $\det(A^\top) = \det(A)$ for any $n \times n$ matrix A, and then we may simply transpose our matrix, apply row operations, and then transpose back at the end.

Corollary 1.7.9 The determinant of an upper triangular matrix is the product of its diagonal entries.

Proof. Let's work with a 3×3 matrix for simplicity. We can repeatedly apply (R1) to an upper triangular matrix A to obtain a diagonal matrix D, where $\det(D) = \det(A)$ by (iv). Then we appeal to parts (i) and (v) from the theorem e.g. denote by $\mathbf{r}_i$ the i^{th} row of the matrix and apply (R3) $\mathbf{r}_1 \leftarrow \lambda_1^{-1}\mathbf{r}_1$ to

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

to obtain

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Then $\det(B_1) = \lambda_1^{-1}\det(D)$ i.e. $\det(D) = \lambda_1 \det(B_1)$. Now we repeat the process:

- Apply $\mathbf{r}_2 \leftarrow \lambda_2^{-1}\mathbf{r}_2$ to B_1 to obtain B_2 so $\det(B_1) = \lambda_2 \det(B_2)$.
- Apply $\mathbf{r}_3 \leftarrow \lambda_3^{-1}\mathbf{r}_3$ to B_2 to obtain $B_3 = I_n$ so $\det(B_2) = \lambda_3 \det(B_3)$.

Combine all of the above to obtain

$$\begin{aligned} \det(A) &= \det(D) \\ &= \lambda_1 \det(B_1) \\ &= \lambda_1\lambda_2 \det(B_2) \\ &= \lambda_1\lambda_2\lambda_3 \det(B_3) \\ &= \lambda_1\lambda_2\lambda_3 \det(I_n) \\ &\overset{\text{(i)}}{=} \lambda_1\lambda_2\lambda_3. \end{aligned}$$

The process above is entirely analogous for any $n \times n$ upper triangular matrix A. ■

Proof of Proposition 1.7.7. Since A is upper triangular, it follows that $A - xI_n$ is also upper triangular and so its determinant is the product of its diagonal entries i.e.

$$\det(A - xI_n) = \prod_{i=1}^{n}(\alpha_{ii} - x)$$

from which we deduce that the eigenvalues of A are the α_{ii}. ■

I've collected some of the main results on diagonalisation below.

- The following theorem is the most general[9] result about diagonalisability.

  Theorem 1.7.10 Let $T\colon V \to V$ be a linear map. Then T is diagonalisable (i.e. there exists a basis E of V with respect to which $[E, T, E]$ is diagonal) if and only if there is a basis of V consisting of eigenvectors of T.

  - Equivalently, an $n \times n$ matrix A over $\mathbb{K}$ is similar to a diagonal matrix iff the space $\mathbb{K}^{n,1}$ has a basis consisting of eigenvectors of A.

- Furthermore, if T is diagonalisable, and $E = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is an ordered basis of eigenvectors of T, and $D = [E, T, E]$, then D is a diagonal matrix and for every $1 \leqslant j \leqslant n$, d_{jj} is the eigenvalue λ_j corresponding to $\mathbf{v}_j$.

[9]The basis need not be orthogonal/orthonormal which is a concept that will be seen shortly.

Verification. The j^{th} column of $[E, T, E]$ consists of the coordinates of $T(\mathbf{v}_j)$ with respect to the basis E i.e. $T(\mathbf{v}_j) = \sum_{i=1}^{n} d_{ij}\mathbf{v}_i = d_{j1}\mathbf{v}_1 + \ldots + d_{jn}\mathbf{v}_n$ and note that $T(\mathbf{v}_j) = \lambda_j \mathbf{v}_j = 0\mathbf{v}_1 + \ldots + 0\mathbf{v}_{j-1} + \lambda_j\mathbf{v}_j + 0\mathbf{v}_{j+1} + \ldots + 0\mathbf{v}_n$ so that

$$d_{j1}\mathbf{v}_1 + \ldots + d_{j,j-1}\mathbf{v}_{j-1} + (d_{jj} - \lambda_j)\mathbf{v}_j + d_{j,j+1}\mathbf{v}_{j+1} + \ldots + d_{j,n}\mathbf{v}_n = \mathbf{0}_V$$

which forces (by linear independence of the basis E) $d_{ij} = 0$ for $i \neq j$, and $d_{ij} = \lambda_j$. ■

- The above theorem doesn't tell you when such a basis exists (only what we can say if it does exist). The following theorem offers a partial confirmation (we really need $r = \dim(V)$ to conclude the existence of a basis!). In particular, it says that if a collection of eigenvalues is distinct, then the corresponding eigenvectors are linearly independent.

Theorem 1.7.11 Let $\lambda_1, \ldots, \lambda_r$ be distinct eigenvalues of $T\colon V \to V$, and denote by $\mathbf{v}_1, \ldots, \mathbf{v}_r$ their corresponding eigenvectors. Then $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly independent.

Proof. Proof by induction on the number of distinct eigenvalues of T.

Base case $(r = 1)$

For $r = 1$, $\mathbf{v}_1 \neq \mathbf{0}_V$ since it's an eigenvector of T, and so $\{\mathbf{v}_1\}$ is a linearly independent set.

Assume the inductive hypothesis

Assume that the theorem holds for $r - 1$ i.e. any $r - 1$ eigenvectors corresponding to $r - 1$ distinct eigenvalues of T are linearly independent.

Inductive step

Suppose that we have r eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ corresponding to the distinct eigenvalues $\lambda_1, \ldots, \lambda_r$. We wish to show that $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a linearly independent set. Suppose that $\alpha_1, \ldots, \alpha_r \in \mathbb{K}$ are such that

$$\alpha_1\mathbf{v}_1 + \ldots + \alpha_r\mathbf{v}_r = \mathbf{0}_V.$$

Apply $T - \lambda_r \operatorname{id}_V$ to both sides of the above equality to "kill[10] off" the eigenvector[11] $\mathbf{v}_r$ and obtain

$$\alpha_1(\lambda_1 - \lambda_r)\mathbf{v}_1 + \ldots + \alpha_{r-1}(\lambda_{r-1} - \lambda_r)\mathbf{v}_{r-1} = \mathbf{0}_V.$$

By our inductive hypothesis, $\{\mathbf{v}_1, \ldots, \mathbf{v}_{r-1}\}$ is linearly independent and so

$$\alpha_1(\lambda_1 - \lambda_r) = \alpha_2(\lambda_2 - \lambda_r) = \ldots = \alpha_{r-1}(\lambda_{r-1} - \lambda_r) = 0.$$

Since the λ_i are distinct, $\lambda_i - \lambda_r \neq 0$ for $i = 1, \ldots, r - 1$ which forces $\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$ and so our original equality at the start reduces to

$$\alpha_r\mathbf{v}_r = \mathbf{0}_V$$

but $\mathbf{v}_r \neq \mathbf{0}_V$ since it's an eigenvector, so $\alpha_r = 0$.

Therefore, $\alpha_i = 0$ for all $i = 1, \ldots, r$ and we conclude that $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a linearly independent set. ■

- What follows is a **sufficient** condition for diagonalisability:

[10]This is a common technique in linear algebra. Removing some part of a linear combination makes solving for the other coefficients easier!

[11]As shall be seen later, we call the kernel $\ker(T - \lambda_r \operatorname{id}_V)$ the *eigenspace* of T with respect to λ_r. The eigenvectors corresponding to λ_r (and the zero vector $\mathbf{0}_V$) are the only elements of this eigenspace.

Corollary 1.7.12 If there are $n = \dim(V)$ distinct eigenvalues of a linear operator $T: V \to V$, then T is diagonalisable.

Proof. Suppose that T has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$. For each i choose an eigenvector $\mathbf{v}_i$ corresponding to λ_i. By the above theorem, $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent, and since $\dim(V) = n$, this set is a basis for V. Thus, T is diagonalisable by the characterisation. ■

The converse is false. It's not true that if T is diagonalisable, then it has n distinct eigenvalues. e.g. the identity map has a single eigenvalue but its matrix representation is clearly diagonal.

1.8 More Diagonalisability

This following three sections are my personal notes as I worked through Sections 5.2 and 5.4 from [1]. Other sources like [2] were also an influence for cross-referencing terminology and proof statements but the main skeleton follows the chronology of [1].

- At the end of the last section, we remarked that if T has $\dim(V)$ distinct eigenvalues, the it's diagonalisable. Also, we found a counterexample to the converse statement; it's not true that if T is diagonalisable then it has $\dim(V)$ distinct eigenvalues.
- However, we do have a partial converse. Namely, if T is diagonalisable, then it imposes a strong condition on the characteristic polynomial — it **splits** into linear factors.

$$\begin{matrix} T\colon V \to V \text{ is a linear operator} \\ \text{whose characteristic} \\ \text{polynomial splits} \end{matrix} \quad \Longleftarrow \quad T\text{-diagonalisable}$$

Definition 1.8.1

- Let $\mathbb{K}[x]$ denote the set of all polynomials in an indeterminate x with coefficients in the field $\mathbb{K}$. That is $\mathbb{K}[x] = \{\alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n : n \in \mathbb{N}_0,\ \alpha_i \in \mathbb{K}\}$. This is a vector space over $\mathbb{K}$ when equipped with polynomial addition and scalar multiplication.
- A polynomial $f(x) \in \mathbb{K}[x]$ **splits over $\mathbb{K}$** if there are scalars $c, a_1, \dots, a_n$ (not necessarily distinct) in $\mathbb{K}$ s.t.

$$f(x) = c(x - a_1)(x - a_2) \cdot \ldots \cdot (x - a_n).$$

Proof. Let T be a diagonalisable linear operator on an n-dimensional vector space V, and let E be an ordered basis for V s.t. $[E, T, E]$ is a diagonal matrix. Let D denote $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$, and let $f(x)$ be the characteristic polynomial of T. Then

$$f(x) = \det(D - xI_n) = \det(\operatorname{diag}(\lambda_1 - x, \dots, \lambda_n - x)) = \prod_{i=1}^{n}(\lambda_i - x) = (-1)^n \prod_{i=1}^{n}(x - \lambda_i).$$

■

It's clear from this theorem that if T is diagonalisable and fails to have $\dim(V)$ distinct eigenvalues, then the characteristic polynomial of T must have repeated roots.

- We can then work in the opposite direction to find a condition that satisfies

$$\begin{matrix} T\colon V \to V \text{ is a linear operator} \\ \text{whose characteristic} \\ \text{polynomial splits} \end{matrix} \quad + \ \text{condition} \quad \Longrightarrow \quad T\text{-diagonalisable}$$

If we can demonstrate this direction, then we'll have another characterisation for the diagonalisability of a linear operator.

'Repetitions' will be of interest so we adopt some terminology:

Definition 1.8.2 Let λ be an eigenvalue of a linear operator (equiv. matrix) with characteristic polynomial $f(x)$. The **(algebraic) multiplicity** of λ is the largest positive integer k for which $(x-\lambda)^k$ is a factor of $f(x)$.

Let T be a diagonalisable linear operator i.e. there exists an ordered basis for V of eigenvectors $E = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of T, and it follows that $[E, T, E]$ is a diagonal matrix whose entries are the eigenvalues of T (including repetitions). In analogous fashion to the proof above, the characteristic polynomial of T is

$$\begin{aligned} f(x) = \det([E,T,E] - xI_n) &= \det(\operatorname{diag}(\lambda_1 - x, \ldots, \lambda_n - x)) \\ &= (-1)^n \prod_{i=1}^{n} (x - \lambda_i) \\ &= (-1)^n (x-\lambda_i)^{e_1} \cdot \ldots \cdot (x-\lambda_n)^{e_n} \end{aligned}$$

where e_i is the algebraic multiplicity of λ_i. It's clear that each eigenvalue λ_i of T must occur as a diagonal entry of $[E,T,E]$ exactly e_i times. Hence, for each eigenvalue λ_i, the basis E contains exactly e_i linearly independent eigenvectors corresponding to λ_i.

Therefore, the number of linearly independent eigenvectors corresponding to a given eigenvalue is of interest in determining whether an operator can be diagonalised.

Definition 1.8.3

- The **eigenspace of T corresponding to an eigenvalue λ** is $\ker(T - \lambda\,\mathrm{id}_V)$.
- The eigenvectors of T corresponding to the eigenvalue λ are the non-zero vectors in the eigenspace of T corresponding to λ.
- The maximum number of linearly independent eigenvectors of T corresponding to λ is the dimension of the eigenspace $\ker(T - \lambda\,\mathrm{id}_V)$ and is called the **geometric multiplicity of λ**.

We now relate this number to the algebraic multiplicity of λ:

Theorem 1.8.4 (**5.7** [1, p. 264]) Let T be a linear operator on a finite-dimensional vector space V, and λ be an eigenvalue of T having algebraic multiplicity m. Then $1 \leqslant \dim(\ker(T - \lambda\,\mathrm{id}_V)) \leqslant m$.

Proof. Choose an ordered basis $(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ of $\ker(T - \lambda\,\mathrm{id}_V)$, and extend it to an ordered basis $E = (\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{v}_{p+1}, \ldots, \mathbf{v}_n)$ for V, and let $A = [E, T, E]$. Observe that $\mathbf{v}_i$ for $i = 1, \ldots, p$ is an eigenvector of T corresponding to λ, and therefore

$$[E,T,E] = \begin{bmatrix} \lambda I_n & B \\ 0 & C \end{bmatrix}.$$

Exercise 1 (Exercise 21 [1, p. 229]) Prove that if $M \in \mathbb{K}^{n,n}$ can be written in the form

$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A and C are square matrices, then $\det(M) = \det(A)\det(C)$.

Proof. by induction on the size of A (a $k \times k$ matrix). The base case with $k = 1$ is represented by

$$M = \begin{bmatrix} a_{11} & b_1 & b_2 & \cdots & b_{n-1} \\ 0 & c_{11} & c_{12} & \cdots & c_{1,n-1} \\ 0 & c_{21} & c_{22} & \cdots & c_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & c_{n-1,1} & c_{n-1,2} & \cdots & c_{n-1,n-1} \end{bmatrix}.$$

Expand out the determinant of M by the first column to get

$$\det(M) = a_{11}\det(C) + 0 + \ldots + 0 = a_{11}\det(C) = \det(A)\det(C).$$

Now assume that the claim holds when A is a $(k-1)\times(k-1)$ matrix. For the inductive step, consider M where A is a $k \times k$ matrix. Expand the determinant of M along the first column:

$$\det(M) = \sum_{j=1}^{k} (-1)^{1+j} m_{j,1} \det(M_{j,1})$$

where $M_{j,1}$ is the minor matrix obtained by removing the j^{th} row and 1^{st} column from M. Notice that the sum only goes up to $j = k$. All terms $m_{j,1}$ for $j > k$ are zero.

Every $M_{j,1}$ for $1 \leqslant j \leqslant k$ is of the form $M_{j,1} = \begin{bmatrix} A_{j,1} & B_j \\ 0 & C \end{bmatrix}$ where:

- $A_{j,1}$ is the $(k-1)\times(k-1)$ matrix obtained from A by removing its j^{th} row and 1^{st} column, and
- and B_j is the $(k-1)\times k$ matrix obtained from B by removing its j^{th} row.

By the inductive hypothesis, each such minor matrix has determinant equal to the product of the determinants of its diagonal blocks in the top left and bottom right. Therefore, our determinant takes the form

$$\begin{aligned}
\det(M) &= \sum_{j=1}^{k} (-1)^{1+j} m_{j,1} \det(M_{j,1}) \\
&= \sum_{j=1}^{k} (-1)^{1+j} a_{j,1} \det(A_{j,1}) \det(C) \\
&= \left(\sum_{j=1}^{k} (-1)^{1+j} a_{j,1} \det(A_{j,1}) \right) \det(C)
\end{aligned}$$

and note that this is $\det(C)$ multiplied by exactly the cofactor expansion of $\det(A)$ by expanding out from the 1^{st} row of A. The claim is proven. ■

By the exercise above, the characteristic polynomial of T is

$$\begin{aligned}
f(x) = \det(A - xI_n) &= \det\left(\begin{bmatrix} (\lambda - x)I_p & B \\ 0 & C - xI_{n-p} \end{bmatrix} \right) \\
&= \det((\lambda - x)I_p)\det(C - xI_{n-p}) \\
&= (\lambda - x)^p g(x) \quad \text{for some polynomial } g(x).
\end{aligned}$$

Therefore, $(\lambda - x)^p$ is a factor of $f(x)$, and hence the algebraic multiplicity of λ is at least p since it may also occur as a root of $g(x)$. However, $\dim(\ker(T - \lambda\,\mathrm{id}_V)) = p$, and so $\dim(\ker(T - \lambda\,\mathrm{id}_V)) \leqslant m$. ■

Example 1.8.5 Let $T\colon \mathbb{K}_2[x] \to \mathbb{K}_2[x]$ be the map defined by $f(x) \longmapsto T(f(x)) = f'(x)$. The matrix of T with respect to the standard ordered basis $(1, x, x^2)$ of $\mathbb{K}_2[x]$ is easy to write down.

- $T(1) = 0 + 0x + 0x^2$ so $[E, 1] = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$.
- $T(x) = 1 + 0x + 0x^2$ so $[E, x] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$.
- $T(x^2) = 0 + 2x + 0x^2$ so $[E, x^2] = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}^\top$.

$$\therefore \; A = [E, T, E] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of T is $f(x) = -x^3$ so T has only one eigenvalue $\lambda_1 = 0$ with algebraic multiplicity 3. Solving $(T - \lambda_1 \operatorname{id}_{\mathbb{K}_2[x]})(\mathbf{v}) = 0$ i.e. solving for $\ker(T)$ i.e. $f'(x) = 0$ tells us that f must be constant. Therefore, $\ker(T - \lambda_1 \operatorname{id}_{\mathbb{K}_2[x]})$ is equal to the subspace of $\mathbb{K}_2[x]$ consisting of constant polynomials. This means that $\{1\}$ is a basis for $\ker(T - \lambda_1 \operatorname{id}_{\mathbb{K}_2[x]})$ and therefore its dimension is 1 i.e. $\lambda_1 = 0$ is the only eigenvalue of T with algebraic multiplicity 3 and geometric multiplicity 1.

- In general, we want a a basis for V consisting of eigenvectors of T (which is an equivalent criterion for T-diagonalisable). In this case, $\dim(V) = \dim(\mathbb{K}_2[x]) = 3$. If we have any hope of T being diagonalisable, then $\lambda_1 = 0$ with algebraic multiplicity 3 must have 3 linearly independent eigenvectors associated with it.
- However, $\dim(\ker(T - \lambda_1 \operatorname{id}_{\mathbb{K}_2[x]})) = 1$.

▶

The following example is slightly less simple that the straightforward eigenspace calculation which reduced to a simple kernel in the first example.

Example 1.8.6 Let $T\colon \mathbb{R}^{3,1} \to \mathbb{R}^{3,1}$ be the linear operator defined by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \longmapsto \begin{bmatrix} 4a_1 + a_3 \\ 2a_1 + 3a_2 \\ a_1 + 4a_3 \end{bmatrix}$$

Let $E = (\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3)$ be the standard ordered basis for $\mathbb{R}^{3,1}$. Then the matrix representation of T with respect to this basis is

$$[E, T, E] = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

and so $\det([E, T, E] - xI_3) = -(x-5)(x-3)^2$. The eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 3$, with algebraic multiplicites 1 and 2 respectively. We can find $\ker(T - \lambda_1 \operatorname{id}_V)$ by computing the nullspace of $A - \lambda_1 I_3$.

In this case, $\mathbb{K} = \mathbb{R}$, $V = \mathbb{K}^{3,1}$ and E is the standard basis of $\mathbb{K}^{3,1}$ so $\Phi_E = \left[\operatorname{id}_{\mathbb{K}^{3,1}}\right]_E^E = \operatorname{id}_{\mathbb{K}^{3,1}}$ and therefore $T = \left[T\right]_E^E$.

$$\begin{tikzcd} V \arrow[r, "T"] & V \\ \mathbb{K}^{3,1} \arrow[u, "\Phi_E"] \arrow[r, "{\left[T\right]_E^E}"'] & \mathbb{K}^{3,1} \arrow[u, "\Phi_E"'] \end{tikzcd}$$

Therefore, $\ker(T - \lambda \operatorname{id}_{\mathbb{K}^{3,1}}) = \ker(\left[T\right]_E^E - \lambda \operatorname{id}_{\mathbb{K}^{3,1}}) = \operatorname{nullspace}([E, T, E] - \lambda I_3)$.

$$\begin{aligned} \ker(T - \lambda_1 \operatorname{id}_{\mathbb{K}^{3,1}}) = \operatorname{nullspace}(A - \lambda_1 I_3) &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^{3,1} : (A - \lambda_1 I_3) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^{3,1} : \begin{bmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \end{aligned}$$

This nullspace is the solution space of the system of linear equations

$$\left\{ \begin{array}{rcrcrcl} -x_1 & & & + & x_3 & = & 0 \\ 2x_1 & - & 2x_2 & + & 2x_3 & = & 0 \\ x_1 & & & - & x_3 & = & 0 \end{array} \right.$$

This can be solved by row-reducing the matrix to obtain the following equivalent system

$$\begin{cases} x_1 & & - & x_3 & = & 0 \\ 2x_1 & x_2 & - & 2x_3 & = & 0 \\ & & & 0 & = & 0 \end{cases}$$

Since x_3 is a free variable, we may let $x_3 = \alpha$ and so the general solution is given by $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Hence, a basis for nullspace$(A - \lambda_1 I_3)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

Similarly, $\ker(T - \lambda_2 \operatorname{id}_{\mathbb{R}^{3,1}}) = \text{nullspace}(A - \lambda_2 I_3)$ is the solution space of the system

$$\begin{cases} x_1 & + & x_3 & = & 0 \\ 2x_1 & + & 2x_3 & = & 0 \\ x_1 & + & x_3 & = & 0 \end{cases} \qquad \text{i.e. } x_1 + x_3 = 0$$

Since the unknown x_2 doesn't appear in any of the equations in this system, we assign it a parametric value say $x_2 = s$, and solve the system for x_1 and x_3, introducing another parameter t. The result is the general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \text{for } s, t \in \mathbb{R}.$$

Since the two vectors are linearly independent, it follows that

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

is a basis for $\ker(T - \lambda_2 \operatorname{id}_{\mathbb{R}^{3,1}})$ and so the geometric multiplicity $\dim(\ker(T - \lambda_2 \operatorname{id}_{\mathbb{R}^{3,1}}))$ of λ_2 is 2.

Thus, we've found that the algebraic multiplicity of each eigenvalue λ_i is equal to its geometric multiplicity. The union of these bases is a linearly independent set and hence a basis for $\mathbb{R}^{3,1}$ consisting of eigenvectors of T. Hence, T is diagonalisable. ▶

The above two examples suggest that the condition we were looking for to complete the implication

$$\underset{\text{polynomial splits}}{\underset{\text{whose characteristic}}{T\colon V \to V \text{ is a linear operator}}} + \text{ condition} \implies T\text{-diagonalisable}$$

is that the algebraic multiplicity of each eigenvalue λ_i is equal to its geometric multiplicity.

Lemma 1.8.7 Let T be a linear operator, and $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T. For each $i = 1, \ldots, k$, let $\mathbf{v}_i \in \ker(T - \lambda_i \operatorname{id}_V)$, the eigenspace corresponding to λ_i. If $\mathbf{v}_1 + \ldots + \mathbf{v}_k = \mathbf{0}_V$, then $\mathbf{v}_i = \mathbf{0}_V$ for all i.

Proof. Suppose that at least one of the $\mathbf{v}_i$ is non-zero. By renumbering if necessary, push the zero vectors to the end so that there exists an m between 1 and k inclusive s.t. $\mathbf{v}_i \neq \mathbf{0}_V$ for all $i \leqslant m$, and $\mathbf{v}_i = \mathbf{0}_V$ for $i > m$. Then for each $i \leqslant m$, $\mathbf{v}_i$ is an eigenvector of T corresponding to λ_i and $\mathbf{v}_1 + \ldots \mathbf{v}_m = \mathbf{0}_V$, but this contradicts **Theorem 1.7.11** which asserts that such $\mathbf{v}_i$ corresponding to distinct eigenvalues λ_i are linearly independent. We conclude that $\mathbf{v}_i = \mathbf{0}_V$ for all i. ■

The following theorem tells us how to construct a linearly independent subset of eigenvectors by collecting bases for the individual eigenspaces.

Theorem 1.8.8 (**5.8** [1]) Let T be a linear operator on a vector space V, and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T. For each i, let S_i be a finite linearly independent subset of the eigenspace $\ker(T - \lambda_i \operatorname{id}_V)$. Then $S = S_1 \cup \ldots \cup S_k$ is a linearly independent subset of V.

Proof. Enumerate each $S_i = \{\mathbf{v}_{i1}, \mathbf{v}_{i2}, \ldots, \mathbf{v}_{in_i}\}$. Then $S = \{\mathbf{v}_{ij} : 1 \leqslant j \leqslant n_i,\ 1 \leqslant i \leqslant k\}$. Consider any scalars $\alpha_{ij} \in \mathbb{K}$ s.t.

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} \alpha_{ij} \mathbf{v}_{ij} = \mathbf{0}_V.$$

For each i, let $\mathbf{w}_i = \sum_{j=1}^{n_i} \alpha_{ij} \mathbf{v}_{ij}$. Then $\mathbf{w}_i \in \ker(T - \lambda_i \operatorname{id}_V)$ for each i, and $\mathbf{w}_1 + \ldots + \mathbf{w}_k = \mathbf{0}_V$. By the previous lemma, this implies that $\mathbf{w}_i = \mathbf{0}_V$ for all $i = 1, \ldots, k$. Since each S_i is a linearly independent set, this forces $\alpha_{ij} = 0$ for all j. Thus, S is a linearly independent set. ■

When is the resulting set a basis for V?

Theorem 1.8.9 (**5.9** [1]) Let T be a linear operator on a finite-dimensional vector space V such that its characteristic polynomial splits. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T. Then

(a) T is diagonalisable iff for all i, the algebraic multiplicity of λ_i equals its geometric multiplicity.

(b) If T is diagonalisable and for each i, E_i is an ordered basis for $\ker(T - \lambda_i \operatorname{id}_V)$, then $E = E_1 \cup \ldots \cup E_k$ is an ordered basis for V consisting of eigenvectors of T.

Proof. Let $\dim(V) = n$. For each i, let e_i denote the algebraic multiplicity of λ_i, and $d_i = \dim(\ker(T - \lambda_i \operatorname{id}_V))$.

(a) " $\Longrightarrow$ "

First suppose that T is diagonalisable i.e. there exists an ordered basis E of eigenvectors of T for V. For each i, let $E_i = E \cap \ker(T - \lambda_i \operatorname{id}_V)$, the set of vectors in E that are eigenvectors corresponding to λ_i, and let n_i denote the number of vectors in E_i. Since E_i is a subset of a subspace of dimension d_i, we know that $n_i \leqslant d_i$. Furthermore, the geometric multiplicity is always between 1 and e_i (inclusive) so $d_i \leqslant e_i$. Since the characteristic polynomial of T splits, the e_i sum to n. Thus,

$$n = \sum_{i=1}^{k} n_i \leqslant \sum_{i=1}^{k} d_i \leqslant \sum_{i=1}^{k} e_i = n.$$

It follows that $\sum_{i=1}^{k} (e_i - d_i) = 0$. Since $d_i \leqslant e_i$ for all i, we conclude that $e_i = d_i$ for all i.

" $\Longleftarrow$ "

Conversely, suppose that $e_i = d_i$ for all i. For each i, let E_i be an ordered basis for $\ker(T - \lambda_i \operatorname{id}_V)$, and let $E = E_1 \cup \ldots \cup E_k$. Theorem 5.8 dictates that E is a linearly independent set. Furthermore, since $e_i = d_i$ for all i, E contains

$$\sum_{i=1}^{k} d_i = \sum_{i=1}^{k} e_i = n$$

vectors. Therefore, E is an ordered basis for V consisting of eigenvectors of T, and we conclude that T is diagonalisable.

■

1.9 Direct Sums

Let $T \colon V \to V$ be a linear operator on a finite-dimensional vector space V. There's a way of decomposing V into simpler subspaces that offer insight into the behaviour of T. In the case of diagonalisable operators, the simpler subspaces are the eigenspaces of the operator.

Definition 1.9.1 The **Minkowski sum** (or sumset) of two subsets A, B of an additive abelian group $(G, +)$ is the set $A + B \coloneqq \{a + b : a \in A,\, b \in B\}$.

Definition 1.9.2 Let $W_1, \ldots, W_k$ be subspaces of a vector space V. The **Minkowski sum** of these subspaces is defined to be the set

$$\{\mathbf{v}_1 + \ldots + \mathbf{v}_k : \mathbf{v}_i \in W_i \text{ for } 1 \leqslant i \leqslant k\}$$

which we denote by $W_1 + \ldots + W_k$ or $\sum_{i=1}^{k} W_i$.

It's simple to verify that the sum of subspaces of a vector space is also a subspace.

Example 1.9.3 Let $V = \mathbb{R}^{3,1}$, W_1 denote the x-y plane, and W_2 the y-z plane. Then $\mathbb{R}^{3,1} = W_1 + W_2$ because any vector

$$\begin{aligned} \mathbb{R}^{3,1} \ni \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ c \end{bmatrix} \in W_1 + W_2 \\ &= \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \in W_1 + W_2 \end{aligned}$$

so there is no unique representation. ▶

We're often interested in sums for which every vector has a unique representation, so we may introduce a condition on a sum $W_1 + W_2$ that guarantees such a representation:

Definition 1.9.4 We call V the **(inner) direct sum of W_1 and W_2** if:

- W_1, W_2 are vector subspaces of V,
- $V = W_1 + W_2$, and
- $W_1 \cap W_2 = \{\mathbf{0}_V\}$.

We denote the direct sum by $V = W_1 \oplus W_2$.

More generally, we may extend this to any finite collection of vector subspaces.

Definition 1.9.5 We call V the **direct sum** of the subspaces $W_1, \ldots, W_k$ and write $V = W_1 \oplus \ldots \oplus W_k$ if V is a sum of the W_i and

$$W_j \cap \sum_{i \neq j} W_i = \{\mathbf{0}_V\} \quad \text{for all } 1 \leqslant j \leqslant k.$$

There are several conditions that are equivalent to the definition of a direct sum:

Theorem 1.9.6 (5.10 [1]) Let $W_1, \ldots, W_k$ be subspaces of a finite-dimensional vector space V. The following conditions are equivalent:

(i) $V = W_1 \oplus \ldots \oplus W_k$

(ii) $V = \sum_{i=1}^{k} W_i$ and for any vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ such that $\mathbf{v}_i \in W_i$: if $\mathbf{v}_1 + \ldots + \mathbf{v}_k = \mathbf{0}_V$, then $\mathbf{v}_i = \mathbf{0}_V$ for all i.

(iii) Each vector $\mathbf{v} \in V$ can be uniquely written as $\mathbf{v} = \mathbf{v}_1 + \ldots + \mathbf{v}_k$ where $\mathbf{v}_i \in W_i$.

(iv) If E_i is an ordered basis for W_i, then $E_1 \cup \ldots \cup E_k$ is an ordered basis for V.

(v) For each i, there exists an ordered basis E_i for W_i such that $E_1 \cup \ldots \cup E_k$ is an ordered basis for V.

In this language, we may rephrase the characterisation of diagonalisability in terms of direct sums:

Theorem 1.9.7 (**5.11** [1]) A linear operator $T\colon V \to V$ on a finite-dimensional vector space V is diagonalisable iff V is the direct sum of the eigenspaces of T.

$$V = \bigoplus_{i=1}^{n} \ker(T - \lambda_i \operatorname{id}_V)$$

1.10 Invariant Subspaces and Cayley-Hamilton

This subsection is really just a bunch of tools for computational ease that arise from the important concept of invariance:

Recall that if $\mathbf{v}$ is an eigenvector of a linear operator $T\colon V \to V$, then T maps the span of $\{\mathbf{v}\}$ into itself. Subspaces that are mapped into themselves are of great important in the study of linear operators.

Definition 1.10.1 A subspace W of V is called a **T-invariant subspace of V** if $T(W) \subseteq W$.

Definition 1.10.2 Let T be a linear operator on V, and let $\mathbf{x} \in V \setminus \{\mathbf{0}_V\}$. The subspace $W = \operatorname{span}(\{\mathbf{x}, T(\mathbf{x}), T^2(\mathbf{x}), \ldots\})$ is called the **T-cyclic subspace of V generated by $\mathbf{x}$**.

- W is T-invariant.
- W is the "smallest" T-invariant subspace of V containing $\mathbf{x}$ i.e. any T-invariant subspace of V containing $\mathbf{x}$ must also contain W.

Cyclic subspaces are used to establish the Cayley-Hamilton theorem.

Theorem 1.10.3 (**5.21** [1]) Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Then the characteristic polynomial of $T|_W$ divides the characteristic polynomial of T.

Proof. Choose an ordered basis $E = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ for W, and extend it to an ordered basis $E' = (\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n)$ for V. Let $A = [E', T, E']$ and $B = [E, T|_W, E]$.

Exercise 2 (Exercise 12, [1, p. 323]) In the proof of Theorem 5.21., prove that

$$A = \begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix}.$$

Proof. Recall that T-invariance of W means that $T(\mathbf{w}) \in W$ for all $\mathbf{w} \in W$. In other words, applying T to a vector in W means the result stays in W.

Since $\mathbf{v}_1, \ldots, \mathbf{v}_k \in W$, $T(\mathbf{v}_i) \in W$ for $i = 1, \ldots, k$. E is a basis for W, so we can express

$$T(\mathbf{v}_i) = \alpha_{1i}\mathbf{v}_1 + \alpha_{2i}\mathbf{v}_2 + \ldots + \alpha_{ki}\mathbf{v}_k$$

for some scalars $\alpha_{1i}, \alpha_{2i}, \ldots, \alpha_{ki} \in \mathbb{K}$. Since E' is a basis for all of V, we may also write

$$T(\mathbf{v}_i) = \alpha_{1i}\mathbf{v}_1 + \ldots + \alpha_{ki}\mathbf{v}_k + 0\mathbf{v}_{k+1} + \ldots + 0\mathbf{v}_n$$

so the coordinates of $T(\mathbf{v}_i)$ with respect to E' are

$$[E', T(\mathbf{v}_i)] = \begin{bmatrix} \alpha_{1i} \\ \vdots \\ \alpha_{ki} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{K}^{n,1}.$$

For $\mathbf{v}_j$ with $j > k$ i.e. $\mathbf{v}_j \in E' \setminus E$, we have no restrictions on the coordinates of $T(\mathbf{v}_j)$ with respect to E' i.e.

$$T(\mathbf{v}_j) = b_{1j}\mathbf{v}_1 + \ldots + b_{kj}\mathbf{v}_k + b_{k+1,j}\mathbf{v}_{k+1} + b_{nj}\mathbf{v}_n \quad \text{for } j > k$$

so

$$[E', T(\mathbf{v}_j)] = \begin{bmatrix} b_{1j} \\ \vdots \\ b_{kj} \\ b_{k+1,j} \\ \vdots \\ b_{n,j} \end{bmatrix} \in \mathbb{K}^{n,1}.$$

Therefore, the matrix $A = [E', T, E']$ is

$$\begin{aligned} A &= \begin{bmatrix} | & & | & | & & | \\ [E', T(\mathbf{v}_1)] & \cdots & [E', T(\mathbf{v}_k)] & [E', T(\mathbf{v}_{k+1})] & \cdots & [E', T(\mathbf{v}_n)] \\ | & & | & | & & | \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} & b_{1,k+1} & \cdots & b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2k} & b_{2,k+1} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} & b_{k,k+1} & \cdots & b_{kn} \\ 0 & 0 & \cdots & 0 & b_{k+1,k+1} & \cdots & b_{k+1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & b_{n,k+1} & \cdots & b_{n,n} \end{bmatrix} \\ &= \begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix} \end{aligned}$$

where $B_1 = [E, T|_W, E]$. ■

Let $f(x)$ be the characteristic polynomial of T and $g(x)$ the characteristic polynomial of $T|_W$. Then

$$\begin{aligned} f(x) &= \det(A - xI_n) \\ &= \det\left(\begin{bmatrix} B_1 - xI_k & B_2 \\ 0 & B_3 - xI_{n-k} \end{bmatrix}\right) \\ &= \underbrace{\det(B_1 - xI_k)}_{g(x)} \det(B_3 - xI_{n-k}) \end{aligned}$$

and so $g(x)$ divides $f(x)$. ■

This theorem tells us that we may use the characteristic polynomial of $T|_W$ to gain information about the characteristic polynomial of T itself. In this regard, cyclic subspaces are useful because the characteristic polynomial of a restriction of a linear operator T to a cyclic subspace is readily computable:

Theorem 1.10.4 (**5.22** [1]) Let T be a linear operator on a finite-dimensional vector space V, and W denote the T-cyclic subspace of V generated by a non-zero vector $\mathbf{v} \in V$. Let $k = \dim(W)$. Then

(a) $\{\mathbf{v}, T(\mathbf{v}), \ldots, T^{k-1}(\mathbf{v})\}$ is a basis for W.

(b) If $a_0\mathbf{v} + a_1T(\mathbf{v}) + \ldots + a_{k-1}T^{k-1}(\mathbf{v}) + T^k(\mathbf{v}) = \mathbf{0}_V$, then the characteristic polynomial of $T|_W$ is

$$f(x) = (-1)^k(a_0 + a_1x + \ldots + a_{k-1}x^{k-1} + x_k).$$

Proof.

(a) Since $\mathbf{v} \neq \mathbf{0}_V$, the set $\{\mathbf{v}\}$ is linearly independent. Let j be the largest positive integer for which $E = \{\mathbf{v}, T(\mathbf{v}), \ldots, T^{j-1}(\mathbf{v})\}$ is a linearly independent collection. Such a j must exist because V is finite-dimensional. Let $Z = \text{span}(E)$. Then E is a basis for Z. Furthermore, the collection $\mathbf{v}, T(\mathbf{v}), \ldots, T^{j-1}(\mathbf{v}), T^j(\mathbf{v})$ is linearly independent i.e. there are scalars for which their linear combination is $\mathbf{0}_V$. Rearrange to write $T^j(\mathbf{v})$ as a linear combination of the vectors in E. Hence, $T^j(\mathbf{v}) \in Z$. We use this information to show that Z is a T-invariant subspace of V.

Let $\mathbf{w} \in Z$. Since $\mathbf{w}$ is a linear combination of the vectors of E, there exist scalars $b_0, b_1, \ldots, b_{j-1}$ s.t.

$$\mathbf{w} = b_0\mathbf{v} + b_1T(\mathbf{v}) + \ldots + b_{j-1}T^{j-1}(\mathbf{v})$$

and hence

$$T(\mathbf{w}) = b_0T(\mathbf{v}) + b_1T^2(\mathbf{v}) + \ldots + b_{j-1}T^j(\mathbf{v})$$

i.e. $T(\mathbf{w})$ is a linear combination of the vectors in E and hence belongs to Z. Thus, Z is T-invariant. Furthermore, $\mathbf{v} \in Z$. Since W is the smallest T-invariant subspace of V that contains $\mathbf{v}$, we have that $W \subseteq Z$. Clearly, $Z \subseteq W$. So we conclude that $Z = W$. It follows that E is a basis for W, and therefore $\dim(W) = j$. Thus, $j = k$.

(b) Now view E (from (a)) as an ordered basis for W. Let $a_0, a_1, \ldots, a_{k-1}$ be scalars s.t.

$$\mathbf{0}_V = a_0\mathbf{v} + a_1T(\mathbf{v}) + \ldots + a_{k-1}T^{k-1}(\mathbf{v}) + T^k(\mathbf{v}).$$

Observe that

$$[E, T|_W, E] = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{bmatrix}$$

The characteristic polynomial of $T|_W$ can be computed by a somewhat laborious cofactor expansion. Denote by $M_k(x; a_0, \ldots, a_{k-1})$ the matrix $A - xI_k$ given by

$$\begin{bmatrix} -x & 0 & \cdots & 0 & -a_0 \\ 1 & -x & \cdots & 0 & -a_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -x - a_{k-1} \end{bmatrix}$$

The characteristic polynomial is

$$\det(A - xI_k) = \det(M_k(x; a_0, \ldots, a_{k-1})).$$

Expand out by the first row (it only has two non-zero terms), denoting by M_{1j} the matrix obtained from M by removing the 1^{st} row and j^{th} column. I'll use M_k as shorthand for $M_k(x; a_0, \ldots, a_{k-1})$.

$$\begin{aligned} \det(A - xI_k) &= (-1)^{1+1}(-x)\det((M_k)_{11}) + (-1)^{1+k}(-a_0)\det((M_k)_{1k}) \\ &= (-x)\det((M_k)_{11}) + (-1)^k a_0 \det((M_k)_{1k}). \end{aligned}$$

The second matrix is easy to compute:

$$(M_k(x; a_0, \ldots, a_{k-1}))_{1k} = \begin{bmatrix} 1 & -x & 0 & \cdots & 0 \\ 0 & 1 & -x & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

is an upper triangular matrix so its determinant is the product of its diagonal entries i.e. $\det((M_k)_{1k}) = 1$.

The slightly more involves determinant is $(M_k)_{11}$. Removing the 1^{st} row and 1^{st} column gives a $(k-1)\times(k-1)$ matrix of the same type as M_k but with its rightmost (final) column of coefficients beginning with a_1 (instead of a_0 as it did with M_k itself):

$$\begin{aligned}(M_k)_{11} &= \begin{bmatrix} -x & 0 & \cdots & 0 & -a_1 \\ 1 & -x & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -x-a_{k-1} \end{bmatrix} \\ &=: M_{k-1}(x; a_1, \ldots, a_{k-1}).\end{aligned}$$

Therefore, we have a recurrence relation

$$\det(M_k(x; a_0, \ldots a_{k-1})) = (-x)\det(M_{k-1}(x; a_1, \ldots, a_{k-1})) + (-1)^k a_0.$$

Performing the recursion once gives the expression above $(k-1)\times(k-1)$ matrix. Performing it twice gives an expression involving a $(k-2)\times(k-2)$ matrix of the same form again. Proceeding inductively, unpacking the recursion $k-1$ times gives an expression with a 1×1 matrix $M_1(x; a_{k-1}) = \begin{bmatrix}-x-a_{k-1}\end{bmatrix}$ whose determinant is simply $-x-a_{k-1}$.

Denote by D_{k,a_0} the determinant $\det(M_k(x; a_0, \ldots, a_{k-1}))$ to save ~~my soul~~ space.

$$\begin{aligned}
D_{k,a_0} &= (-x)D_{k-1,a_1} + (-1)^k a_0 \\
&= (-x)\left((-x)D_{k-2,a_2} + (-1)^{k-1}a_1\right) + (-1)^k a_0 \\
&= (-x)^2 D_{k-2,a_2} + (-1)^{k-1}(-x)a_1 + (-1)^k a_0 \\
&= (-x)^2\left((-x)D_{k-3,a_3} + (-1)^{k-2}a_2\right) + (-1)^{k-1}(-x)a_1 + (-1)^k a_0 \\
&= (-x)^3 D_{k-3,a_3} + (-1)^{k-2}(-x)^2 a_2 + (-1)^{k-1}(-x)a_1 + (-1)^k a_0 \\
&= \ldots \\
&= (-x)^{k-1}D_{k-1,a_{k-1}} + (-1)^2(-x)^{k-2}a_{k-2} + \ldots + (-1)^{k-2}(-x)^2 a_2 \\
&\qquad + (-1)^{k-2}(-x)^2 a_2 + (-1)^{k-1}(-x)a_1 + (-1)^k a_0 \\
&= (-x)^{k-1}\det\left(\begin{bmatrix}-x-a_{k-1}\end{bmatrix}\right) + \sum_{i=0}^{k-2}(-1)^{k-i}(-x)^i a_i \\
&= (-x)^k + (-x)^{k-1}a_{k-1} + \sum_{i=0}^{k-2}(-1)^k x^i a_i \\
&= (-1)^k x^k + (-1)^{k-1}x^{k-1}a_{k-1} + \sum_{i=0}^{k-2}(-1)^k x^i a_i \\
&= (-1)^k(a_0 + a_1 x + a_2 x^2 + \ldots + a_{k-1}x^{k-1} + x^k)
\end{aligned}$$

as desired.

■

As an illustration of the importance of the above theorem, we prove the Cayley-Hamilton theorem:

Theorem 1.10.5 (Cayley-Hamilton) Let $T\colon V \to V$ be a linear operator on a finite-dimensional space V, and let $f(x)$ be the characteristic polynomial of T. Then $f(T) = T_0$, the zero-transformation. That is, T "satisfies" its characteristic equation.

Proof. We show that $f(T)(\mathbf{v}) = \mathbf{0}_V$ for all $\mathbf{v} \in V$. This is obvious for $\mathbf{v} = 0_V$ because $f(T)$ is linear. Suppose that $\mathbf{v} \neq \mathbf{0}_V$. Let W be the T-cyclic subspace of V generated by $\mathbf{v}$, and suppose that $\dim(W) = k$. By the previous theorem's part (a), $E = (\mathbf{v}, T(\mathbf{v}), \ldots, T^{k-1}(\mathbf{v}))$ is a basis for

W and so we may write $T^k(\mathbf{v})$ as a linear combination of the vectors in E i.e. there exist scalars $a_0, a_1, \ldots, a_{k-1}$ s.t.

$$a_0\mathbf{v} + a_1T(\mathbf{v}) + \ldots + a_{k-1}T^{k-1}(\mathbf{v}) + T^k(\mathbf{v}) = \mathbf{0}_V.$$

Part (b) from the same theorem tells us that

$$g(x) = (-1)^k(a_0 + a_1x + \ldots + a_{k-1}x^{k-1} + x^k)$$

is the characteristic polynomial of $T|_W$. Combining these two equations yields

$$\begin{aligned} g(T)(\mathbf{v}) &= (-1)^k(a_0 \operatorname{id}_V + a_1T + \ldots + a_{k-1}T^{k-1} + T^k)(\mathbf{v}) \quad \text{by the first equation} \\ &= \mathbf{0}_V \quad \text{by the second equation.} \end{aligned}$$

Furthermore, $g(x)$ divides the characteristic polynomial $f(x)$ of T so there exists some polynomial $q(x)$ s.t. $f(x) = q(x)g(x)$ and so

$$\begin{aligned} f(T)(\mathbf{v}) &= (q(T) \circ g(T))(\mathbf{v}) \\ &= q(T)(g(T)(\mathbf{v})) \\ &= q(T)(\mathbf{0}_V) \\ &= \mathbf{0}_V. \end{aligned}$$

■

Corollary 1.10.6 (Cayley-Hamilton for Matrices) Let A be an $n \times n$ matrix, and let $f(x)$ be the characteristic polynomial of A. Then $f(A) = 0_{n\times n}$.

CHAPTER 2

Bilinear Algebra

The mathematical structure that will play host to the idea of orthogonality is "Euclidean space" — a pair of a real vector space, and a particular type of "bilinear" map that encodes information about angles and lengths. We'll define all these terms posthaste.

2.1 Bilinear Maps

Definition 2.1.1 Let V and W be vector spaces over a field $\mathbb{K}$. A **bilinear map on V and W** is a map $\tau\colon V \times W \to \mathbb{K}$ satisfying the following two properties

1. Linearity in the first entry:
$$\tau(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2, \mathbf{w}) = \alpha\tau(\mathbf{v}_1, \mathbf{w}) + \beta\tau(\mathbf{v}_2, \mathbf{w}).$$
2. Linearity in the second entry:
$$\tau(\mathbf{v}, \alpha \mathbf{w}_1 + \beta \mathbf{w}_2) = \alpha\tau(\mathbf{v}, \mathbf{w}_1) + \beta\tau(\mathbf{v}, \mathbf{w}_2).$$

for all $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in V$, $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2 \in W$, and $\alpha, \beta \in \mathbb{K}$.

2.1.1 Matrix Representation

Let $\tau\colon V \times W \to \mathbb{K}$ be a bilinear map, and choose bases $E = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ of V, and $F = (\mathbf{f}_1, \ldots, \mathbf{f}_m)$ of W. For any $\mathbf{v} \in V$, $\mathbf{w} \in W$, we may uniquely write $\mathbf{v} = x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n$ and $\mathbf{w} = y_1\mathbf{f}_1 + \ldots + y_m\mathbf{f}_m$. By the bilinearity of τ, we obtain

$$\begin{aligned}\tau(\mathbf{v}, \mathbf{w}) = \tau\left(\sum_{i=1}^{n} x_i\mathbf{e}_i, \sum_{j=1}^{m} y_j\mathbf{f}_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i\tau(\mathbf{e}_i, \mathbf{f}_j)y_j &=: [F, \mathbf{w}]^\top A[E, \mathbf{v}] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} y_j\alpha_{ji}x_i\end{aligned}$$

where $A = (\alpha_{ij})$ is an $m \times n$ matrix whose $(i,j)^{\text{th}}$ entry is (by comparing the summations) $\alpha_{ij} = \tau(\mathbf{e}_j, \mathbf{f}_i)$.

Definition 2.1.2 In alignment with the above discussion, the matrix $A = (\alpha_{ij}) \in \mathbb{K}^{m,n}$ is the **matrix of the bilinear map $\tau\colon V \times W \to \mathbb{K}$ with respect to the bases E of V and F of W**.

Theorem 2.1.3 Let V and W be vector spaces over $\mathbb{K}$ of dimensions n and n respectively. Then, for a given choice of bases of V and W, there is a one-to-one correspondence between the set $\mathrm{Bil}_{\mathbb{K}}(V \times W; \mathbb{K})$ of bilinear maps $\tau\colon V \times W \to \mathbb{K}$, and the set $\mathbb{K}^{m,n}$ of $m \times n$ matrices over $\mathbb{K}$.

$$\mathrm{Bil}_{\mathbb{K}}(V \times W; \mathbb{K}) \cong \mathbb{K}^{m,n}$$

2.1.2 Change of Basis (Bilinear Map)

Let A be the matrix of the bilinear map $\tau\colon W \times V \to \mathbb{K}$ with respect to the bases E of V and F of W, and B be its matrix with respect to the bases E' of V and F' of W. Consider the change-of-basis matrices $P = [E', \mathrm{id}_V, E]$ from E to E', and $Q = [F', \mathrm{id}_V, F]$ from F to F'. Then for any $\mathbf{w} \in W$ and $\mathbf{v} \in V$:

$$\begin{aligned}[F,\mathbf{w}]^\top A[E,\mathbf{v}] = \tau(\mathbf{v},\mathbf{w}) &= [F',\mathbf{w}]^\top B[E,\mathbf{v}] \\ &= ([F',\mathrm{id}_W,F][F,\mathbf{w}])^\top B([E',\mathrm{id}_V,E][E,\mathbf{v}]) \\ &= [F,\mathbf{w}]^\top [F',\mathrm{id}_W,F]^\top B[E',\mathrm{id}_V,E]\,[E,\mathbf{v}].\end{aligned}$$

Both matrices give rise to the same bilinear map. By **Theorem 2.1.3**, once bases of W and V are fixed, the matrix representation is unique so $A = Q^\top BP$ which may be re-written as

$$B = (Q^\top)^{-1}AP^{-1}$$

where $Q = [F',\mathrm{id}_W,F]$ and $P = [E',\mathrm{id}_V,E]$.

In an analogous sense to the linear case, the matrices A and B representing the bilinear map with respect to different bases are called equivalent. Again, since P and Q are invertible, the definition in general is streamlined to:

Definition 2.1.4 Two matrices A and B are called **equivalent** (*in the sense of representing bilinear maps*) if there exist invertible matrices P and Q s.t. $B = Q^\top AP$.

If we instead focus on bilinear forms $\tau\colon V\times V \to \mathbb{K}$, then we simply say the matrix representation is with respect to the chosen basis E of V.

Theorem 2.1.5 Let A be the matrix representation of a bilinear form τ on V with chosen basis E, and B the matrix of τ with respect to a basis E' of V. Consider the change-of-basis matrix $P = [E',\mathrm{id}_V,E]$ from E to E'. Then $B = (P^\top)^{-1}AP^{-1}$. We say that A and B are congruent, and more generally that:

Definition 2.1.6 Two square matrices A and B are called **congruent** if there exists an invertible matrix P s.t. $B = P^\top AP$.

I'm a great subscriber to the consistency of representing the old basis with a letter e.g. E, and the new basis as E', and defining the change-of-basis matrices from the "old" to the "new" bases e.g. $P = [E',\mathrm{id}_U,E]$.

Table 2.1: A table to take stock of the different matrix comparisons seen thus far.

Type of Map	**Bases (Old/New)**	**Property**	**Change-of-Basis Matrices**	**Expression**	
Linear Map $T\colon U\to V$	E of U, F of V E' of U, F' of V	A and B are equivalent	$P=[E',\mathrm{id}_U,E]$ $Q=[F',\mathrm{id}_V,F]$	$B=QAP^{-1}$	$B=QAP$
Linear Operator $T\colon V\to V$	E of V E' of V	A and B are similar	$P=[E',\mathrm{id}_V,E]$	$B=PAP^{-1}$	$B=P^{-1}AP$
Bilinear Map $T\colon V\times W\to\mathbb{K}$	E of V, F of W E' of V, F' of W	A and B are equivalent	$P=[E',\mathrm{id}_V,E]$ $Q=[F',\mathrm{id}_W,F]$	$B=(Q^\top)^{-1}AP^{-1}$	$B=Q^\top AP$
Bilinear Form $T\colon V\times V\to\mathbb{K}$	E of V, E' of V	A and B are congruent	$P=[E',\mathrm{id}_V,E]$	$B=(P^\top)^{-1}AP^{-1}$	$B=P^\top AP$

The final column at the end of the table represents the more elegant form one usually encounters in books, where authors often pick and choose the directions of the change of bases to hide inverses e.g. my $B = QAP^{-1}$ becomes $B = QAP$. I prefer to stick to the old-to-new convention.

2.2 Bilinear Forms

Definition 2.2.1 We give the name **bilinear form** to a bilinear map on V and W when $V = W$ i.e. a bilinear map $\tau\colon V \times V \to \mathbb{K}$.

2.2.1 Symmetric bilinear form

We encounter a particular example of a symmetric bilinear form on $\mathbb{R}^n$ when using the scalar product of two real vectors to measure how "similar" they are.

Definition 2.2.2 A **bilinear form τ on V is called symmetric** if $\tau(\mathbf{v}, \mathbf{w}) = \tau(\mathbf{w}, \mathbf{v})$ for all $\mathbf{v}, \mathbf{w} \in V$. An $n \times n$ matrix is called symmetric if $A^\top = A$.

Proposition 2.2.3 Let E be a finite basis of a vector space V. Then, a bilinear form τ on V is symmetric iff its matrix representation A w.r.t. E is symmetric.

Proof.

$\Longrightarrow$ For the forward implication, let τ be symmetric. The matrix A representing τ with respect to $E = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ has $(i, j)^{\text{th}}$ entry $\alpha_{ij} = \tau(\mathbf{e}_j, \mathbf{e}_i)$. By symmetry of τ, $\alpha_{ij} := \tau(\mathbf{e}_j, \mathbf{e}_i) = \tau(\mathbf{e}_i, \mathbf{e}_j) =: \alpha_{ji}$. This demonstrates that A is symmetric.

$\Longleftarrow$ For the converse, suppose that $A^\top = A$. We wish to show, as a consequence, that $\tau(\mathbf{v}, \mathbf{w}) = \tau(\mathbf{w}, \mathbf{v})$ for any $\mathbf{v}, \mathbf{w} \in V$. We may uniquely write $\mathbf{v} = x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n$ and $\mathbf{w} = y_1\mathbf{e}_1 + \ldots + y_n\mathbf{e}_n$ whose coordinate vectors are

$$[E, \mathbf{v}] = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad [E, \mathbf{w}] = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

By **Definition 2.1.2**, our expression for $\tau(\mathbf{v}, \mathbf{w})$ can be written in terms of the coordinate vectors above:

$$\begin{aligned}
\tau(\mathbf{v}, \mathbf{w}) = [E, \mathbf{w}]^\top A [E, \mathbf{v}] &= \sum_{i,j=1}^{n} y_j \alpha_{ji} x_i \\
&= \sum_{j,i=1}^{n} y_i \alpha_{ij} x_j \quad \text{by swapping the variables of summation} \\
&= \sum_{j,i=1}^{n} x_j \alpha_{ji} y_i \\
&= \sum_{i,j=1}^{n} x_j \alpha_{ji} y_i \quad \text{since } A^\top = A \\
&= [E, \mathbf{v}]^\top A [E, \mathbf{w}] \\
&= \tau(\mathbf{v}, \mathbf{w}).
\end{aligned}$$

■

Let $A = I_n \in \mathbb{R}^{n,n}$. The corresponding symmetric bilinear form is called:

Example 2.2.4 The **standard inner product on $\mathbb{R}^{n,1}$** is defined for $\mathbf{x}, \mathbf{y}$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^{n} x_i y_i = \mathbf{x}^\top \mathbf{y}.$$

▶

2.3 Sesquilinear Maps and Forms

In preparation for the next subsection which introduces a type of function (the inner product on a vector space over $\mathbb{K}$) that generalises the standard inner product on $\mathbb{R}^n$, we define the class of maps (sesquilinear forms) for which the inner product is a special case. Slightly more generally, we define:

Definition 2.3.1 A **sesquilinear map on V and W** is a map $\varsigma\colon V \times W \to \mathbb{C}$ satisfying the following two properties

- Linearity in the first entry

$$\varsigma(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2, \mathbf{w}) = \alpha\varsigma(\mathbf{v}_1, \mathbf{w}) + \beta\varsigma(\mathbf{v}_2, \mathbf{w})$$

- Conjugate linearity in the second[1] entry

$$\varsigma(\mathbf{v}, \alpha\mathbf{w}_1 + \beta\mathbf{w}_2) = \overline{\alpha}\varsigma(\mathbf{v}, \mathbf{w}_1) + \overline{\beta}\varsigma(\mathbf{v}, \mathbf{w}_2)$$

for all $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in V$, $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2 \in W$, and $\alpha, \beta \in \mathbb{K}$.

Corollary 2.3.2 In analogous fashion to the description of bilinear maps, let $E = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ be a basis for V, and $F = (\mathbf{f}_1, \dots, \mathbf{f}_n)$ be a basis for W. For any $\mathbf{v} \in V$ and $\mathbf{w} \in W$, we may uniquely write

$$\mathbf{v} = x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n \quad \text{and} \quad \mathbf{w} = y_1\mathbf{f}_1 + \ldots + y_m\mathbf{f}_m,$$

and by the sesquilinearity of ς we obtain

$$\begin{aligned}\varsigma(\mathbf{v}, \mathbf{w}) = \tau\left(\sum_{i=1}^{n} x_i\mathbf{e}_i, \sum_{j=1}^{m} y_j\mathbf{f}_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i\varsigma(\mathbf{e}_i, \mathbf{f}_j)\overline{y_j} &=: [F, \mathbf{w}]^\top A[E, \mathbf{v}] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \overline{y_j}\alpha_{ji}x_i\end{aligned}$$

from which we conclude that the matrix of the sesquilinear map $\varsigma\colon V \times W \to \mathbb{K}$ with respect to the bases E of V and F of W is $A \in \mathbb{K}^{m,n}$ defined by $\alpha_{ij} = \varsigma(\mathbf{e}_j, \mathbf{f}_i)$ and satisfies the above expression for all $\mathbf{v} \in V$, $\mathbf{w} \in W$.

Definition 2.3.3 We give the name **sesquilinear form** to a sesquilinear map on V and W when $V = W$ i.e. a sesquilinear map $\varsigma\colon V \times V \to \mathbb{K}$.

[1]The prefix 'sesqui-' means 'one and a half' which is a reference to the "half" linearity in the second entry.

2.4 Inner Products

Inner products generalise the dot (or scalar) product on $\mathbb{R}^n$. An inner product encodes information about the geometry of a vector space like the length of a vector, the angles between vectors, and so allows us to make comments on whether vectors are orthogonal (the generalisation of perpendicularity in $\mathbb{R}^2$), and define closest points (as typically found in optimisation problems). From a more abstract perspective, an inner product allows one to identify a vector space with its dual. All of these ideas will be covered in this section.

Definition 2.4.1 An inner product on a vector space V defined over $\mathbb{K}$ (which[2] can be $\mathbb{R}$ or $\mathbb{C}$) is a map $\langle\cdot,\cdot\rangle\colon V\times V\to\mathbb{K}$ satisfying the following properties:

1. Linearity in the first entry: $\langle\lambda\mathbf{x}+\mu\mathbf{y},\mathbf{z}\rangle=\lambda\langle\mathbf{x},\mathbf{z}\rangle+\mu\langle\mathbf{y},\mathbf{z}\rangle$.
2. Conjugate symmetry: $\langle\mathbf{x},\mathbf{y}\rangle=\overline{\langle\mathbf{y},\mathbf{x}\rangle}$.
3. Non-negativity: $\mathbb{R}\ni\langle\mathbf{x},\mathbf{x}\rangle\geqslant 0$ for all $\mathbf{x}$.
4. Separates points: $\langle\mathbf{x},\mathbf{x}\rangle=0$ iff $\mathbf{x}=\mathbf{0}_V$.

Remarks 2.4.2

- Let $\mathbb{K}=\mathbb{C}$. For any $\mathbf{x},\mathbf{y},\mathbf{z}\in V$, and any scalars $\alpha,\beta\in\mathbb{C}$, we have a special kind of linearity in the second entry of a complex inner product $\langle\cdot,\cdot\rangle$ — **conjugate linearity**:

$$\begin{aligned}\langle\mathbf{x},\alpha\mathbf{y}+\beta\mathbf{z}\rangle&=\overline{\langle\alpha\mathbf{y}+\beta\mathbf{z},\mathbf{x}\rangle}&&\text{by complex conjugate symmetry}\\&=\overline{\alpha\langle\mathbf{y},\mathbf{x}\rangle+\beta\langle\mathbf{z},\mathbf{x}\rangle}&&\text{by linearity in the first coordinate}\\&=\overline{\alpha}\,\overline{\langle\mathbf{y},\mathbf{x}\rangle}+\overline{\beta}\,\overline{\langle\mathbf{z},\mathbf{x}\rangle}&&\text{since complex conjugation is distributive}\\&=\overline{\alpha}\langle\mathbf{x},\mathbf{y}\rangle+\overline{\beta}\langle\mathbf{x},\mathbf{z}\rangle.\end{aligned}$$

- In the case that $\mathbb{K}=\mathbb{R}$ i.e. V is a real vector space, we have a real inner product and:
  - complex conjugate symmetry reduces to symmetry: $\langle\mathbf{x},\mathbf{y}\rangle=\langle\mathbf{y},\mathbf{x}\rangle$ for all $\mathbf{x},\mathbf{y}\in V$,
  - and so conjugate linearity reduces to linearity in the second entry.

  So we can say that any inner product $\langle\cdot,\cdot\rangle\colon V\times V\to\mathbb{R}$ on a vector space over $\mathbb{R}$ is a **symmetric bilinear form** on V that is **also** non-negative and separates points.

The following fact is used so many times throughout so I'm highlighting it (in post) as important.

Lemma 2.4.3 Suppose that $\langle\mathbf{x},\mathbf{y}-\mathbf{z}\rangle=0$ for all $\mathbf{x},\mathbf{y},\mathbf{z}\in V$. Then $\mathbf{y}=\mathbf{z}$.

Proof. Rearrange by linearity in the first entry to obtain $\langle\mathbf{x},\mathbf{y}-\mathbf{z}\rangle=0$ for all $\mathbf{x}\in V$. In particular, let $\mathbf{x}=\mathbf{y}-\mathbf{z}$, so $\langle\mathbf{y}-\mathbf{z},\mathbf{y}-\mathbf{z}\rangle=0\iff\mathbf{y}-\mathbf{z}=\mathbf{0}_V$ i.e. $\mathbf{y}=\mathbf{z}$. ■

Example 2.4.4 The **standard inner product on $\mathbb{K}^{n,1}$** is defined for $\mathbf{x},\mathbf{y}$ by

$$\langle\mathbf{x},\mathbf{y}\rangle:=\sum_{i=1}^{n}x_i\,\overline{y_i}=\overline{\mathbf{y}}^{\top}\mathbf{x}.$$

We may view this as a sesquilinear form on $\mathbb{K}^{n,1}$ whose matrix representation is I_n. ▶

[2] I believe $\mathbb{K}$ needs to have an ordered sub-field in order to define non-negativity and this pretty much leaves $\mathbb{R}$ or $\mathbb{C}$ as the only candidates.

Definition 2.4.5 The pair $(V, \langle\cdot,\cdot\rangle)$ of a vector space V over $\mathbb{R}$, and a real inner product is called a **Euclidean space**.

Euclidean spaces serve as a structure that connects many concrete ideas (like geometry) to the abstractness of vector spaces.

The generalisation of perpendicularity to an arbitrary vector space equipped with a bilinear form is called orthogonality.

Definition 2.4.6 Let τ be a bilinear form on a vector space V over $\mathbb{K}$.

- Two vectors $\mathbf{v}, \mathbf{w} \in V$ are said to be **orthogonal** (with respect to τ) if $\tau(\mathbf{v}, \mathbf{w}) = 0$.
- A collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots$ in V is said to be:
  - orthogonal if it's pairwise orthogonal i.e. for any $\mathbf{v}_i, \mathbf{v}_j \in V$, we have that $\tau(\mathbf{v}_i, \mathbf{v}_j) = 0$.
  - **orthonormal** if it's orthogonal and every vector is a unit vector

$$\text{i.e. } \{\mathbf{v}_1, \mathbf{v}_2, \ldots\} \text{ is orthonormal } \iff \tau(\mathbf{v}_i, \mathbf{v}_j) = \delta_{ij}.$$

Definition 2.4.7 Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space. A subset of V is called an **orthonormal basis for V** if it is an ordered basis for V that is also orthonormal.

Corollary 2.4.8 If $E = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is an orthonormal basis for an inner product space V, and τ is a bilinear form on V, then the matrix representation $A = (\alpha_{ij})$ of τ with respect to E is the identity matrix.

Proof. Observe that $\alpha_{ij} = \tau(\mathbf{e}_j, \mathbf{e}_i) = \delta_{ij}$ since E is orthonormal. ■

Example 2.4.9 The standard ordered basis $(\mathfrak{e}_1, \ldots, \mathfrak{e}_n)$ for $\mathbb{K}^{n,1}$ is an orthonormal basis for $\mathbb{K}^{n,1}$.
▶

Example 2.4.10 Let $(V, \langle\cdot,\cdot\rangle)$ be a finite-dimensional inner product space over $\mathbb{K}$. Let $E = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be an orthonormal basis of V. Let $\mathbf{v}, \mathbf{w} \in V$. Since E is a basis, we may uniquely write these vectors as $\mathbf{v} = x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n$ and $\mathbf{w} = y_1\mathbf{e}_1 + \ldots + y_n\mathbf{e}_n$. We may compute the inner product of these two vectors

$$\begin{aligned}
\langle \mathbf{v}, \mathbf{w}\rangle &= \left\langle \sum_{i=1}^n x_i\mathbf{e}_i, \sum_{j=1}^n y_j\mathbf{e}_j \right\rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n x_i\overline{y_j}\langle \mathbf{e}_i, \mathbf{e}_j\rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n x_i\overline{y_j}\delta_{ij} \\
&= \sum_{i=1}^n x_i\overline{y_j} \\
&= \overline{[E, \mathbf{w}]}^\top [E, \mathbf{v}] \\
&= \langle [E, \mathbf{v}], [E, \mathbf{w}]\rangle_{\mathbb{K}^{n,1}}
\end{aligned}$$

which is the standard inner product of their coordinate vectors in $\mathbb{K}^{n,1}$.

▶

2.4.1 Inner products encode length

Any inner product $\langle\cdot,\cdot\rangle$ on V defines a norm on V, denoted by $\|\cdot\|\colon V\to\mathbb{R}$ and defined for any $\mathbf{x}\in V$ by $\|\mathbf{x}\|=\sqrt{\langle\mathbf{x},\mathbf{x}\rangle}$.

Definition 2.4.11 Let V be a vector space over a field $\mathbb{K}$. A **norm on V** is a map $\|\cdot\|\colon V\to\mathbb{R}$ that satisfies the following properties:

- Non-negativity: For every $\mathbf{x}\in V$, $\|\mathbf{x}\|\geqslant 0$.
- Separation of points: $\|\mathbf{x}\|=0 \iff \mathbf{x}=\mathbf{0}_V$.
- Absolute homogeneity: For any $\mathbf{x}\in V$, and any scalar $\alpha\in\mathbb{K}$, we have that $\|\alpha\mathbf{x}\|=|\alpha|\|\mathbf{x}\|$.
- Triangle inequality (also called sub-additivity): For any $\mathbf{x},\mathbf{y}\in V$, $\|\mathbf{x}+\mathbf{y}\|\leqslant\|\mathbf{x}\|+\|\mathbf{y}\|$.

The pair $(V,\|\cdot\|)$ is called a **normed vector space**.

The remainder of this section covers my personal interpretation of the salient points from Chapter 6 of [1] — Friedberg, Insel, and Spence's *Linear Algebra (4th Edition)*.

2.4.2 Gram-Schmidt Orthogonalisation

Theorem 2.4.12 Given an orthogonal subset $S=\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ of V consisting of non-zero vectors, if $\mathbf{y}\in\operatorname{span}(S)$ then

$$\mathbf{y}=\sum_{i=1}^{k}\frac{\langle\mathbf{y},\mathbf{v}_i\rangle}{\|\mathbf{v}_i\|^2}\mathbf{v}_i.$$

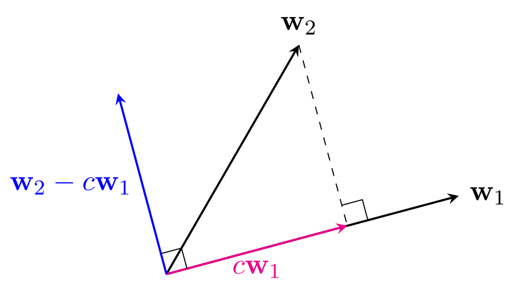
Corollary 2.4.13 If, in addition, S is orthonormal, then $\mathbf{y}=\displaystyle\sum_{i=1}^{k}\langle\mathbf{y},\mathbf{v}_i\rangle\mathbf{v}_i$.

Corollary 2.4.14 An orthogonal subset consisting of non-zero vectors of an inner product space V is linearly independent.

Theorem 2.4.15 Every finite-dimensional vector space admits an orthonormal basis.

The above fact follows from the next theorem.

Motivation: The following diagram is in $\mathbb{R}^2$ and the notion of orthogonality presents itself visually as perpendicularity (but this perspective generalises without difficulty to an abstract finite-dimensional vector space V over $\mathbb{K}$ equipped with an inner product). Consider a linearly independent subset $\{\mathbf{w}_1,\mathbf{w}_2\}$ of V (and hence a basis for some two-dimensional subspace). We wish to construct an orthogonal set from $\{\mathbf{w}_1,\mathbf{w}_2\}$ that spans the same subspace.



The diagram suggests that if we subtract the component of $\mathbf{w}_2$ that is directed parallel to $\mathbf{w}_1$, then we'll arrive at a vector $\mathbf{w}_2 - c\mathbf{w}_1$ orthogonal to $\mathbf{w}_1$. The vector that achieves this will satisfy

$$\begin{aligned} 0 = \langle \mathbf{w}_2 - c\mathbf{w}_1, \mathbf{w}_1 \rangle &= \langle \mathbf{w}_2, \mathbf{w}_1 \rangle - c\langle \mathbf{w}_1, \mathbf{w}_1 \rangle \quad \text{by linearity} \\ &= \langle \mathbf{w}_2, \mathbf{w}_1 \rangle - c\|\mathbf{w}_1\|^2 \end{aligned}$$

from which we conclude that

$$c = \frac{\langle \mathbf{w}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}.$$

This coefficient c is called **the projection coefficient of $\mathbf{w}_2$ on $\mathbf{w}_1$** and measures how much of $\mathbf{w}_2$ runs in the direction of $\mathbf{w}_1$. Similarly, the inner product $\langle \mathbf{v}, \mathbf{x} \rangle$ may be thought of loosely as how much $\mathbf{v}$ runs in the direction of $\mathbf{x}$ (or how much of a shadow the component $\mathbf{v}$ leaves on the span of the one-dimensional subspace spanned by $\mathbf{x}$).

Thus, we've constructed the vector

$$\mathbf{v}_2 := \mathbf{w}_2 - c\mathbf{w}_1 = \mathbf{w}_2 - \frac{\langle \mathbf{w}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1$$

which is orthogonal to $\mathbf{w}_1$. Does $\{\mathbf{w}_1, \mathbf{v}_2\}$ have the same span as $\{\mathbf{w}_1, \mathbf{w}_2\}$? Yes! We state the extension of this result to any finite linearly independent subset:

Theorem 2.4.16 Let V be an inner product space and $S = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ be a linearly independent subset of V. Define $S' = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ by $\mathbf{v}_1 = \mathbf{w}_1$, and define $\mathbf{v}_k$ inductively by

$$\mathbf{v}_k = \mathbf{w}_k - \sum_{j=1}^{k-1} \frac{\langle \mathbf{w}_k, \mathbf{v}_j \rangle}{\|\mathbf{v}_j\|^2}\mathbf{v}_j \quad \text{for } k = 2, \dots, n.$$

Then S' is an orthogonal set of non-zero vectors in V s.t.

$$\operatorname{span}(\mathbf{w}_1, \dots, \mathbf{w}_k) = \operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$$

for every $k = 1, \dots, n$. ■

The construction of $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ in the theorem above is called the **Gram-Schmidt orthogonalisation process**.

Proof of Theorem 2.4.15. If we let E_0 be an ordered basis for V (which exists because V is finite-dimensional), then we can apply the Gram-Schmidt process to obtain an orthogonal set E' of non-zero vectors with $\operatorname{span}(E') = \operatorname{span}(E_0) = V$. By normalising each vector in E', we obtain an orthonormal set E that generates V. By **Corollary 2.4.14**, E is linearly independent, and so E is an orthonormal basis for V. ■

Let V be a finite-dimensional inner product space. By the above theorem, it certainly admits an orthonormal basis $E = (\mathbf{v}_1, \dots, \mathbf{v}_n)$. Then we may represent any $x \in V$ as a linear combination of the vectors in E i.e.

$$\mathbf{x} = \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i.$$

Also, this gives us a simple way to compute the entries of the matrix representation of a linear operator $T\colon V \to V$ with respect to an orthonormal basis E. Let $A = (a_{ij})_{ij} = [E, T, E]$. Its $(i, j)^{\text{th}}$ entry is $a_{ij} = \langle T(\mathbf{v}_j), \mathbf{v}_i \rangle$.

Proof. From **Theorem 2.4.15**, we have $T(\mathbf{v}_j) = \sum_{i=1}^{n} \langle T(\mathbf{v}_j), \mathbf{v}_i \rangle \mathbf{v}_i$. ■

2.4.3 Orthogonal Complement

For problems of the type of finding the distance in $V = \mathbb{R}^3$ of a point P from a plane $W \subseteq \mathbb{R}^3$, the notion of the orthogonal complement of a subset of V finds a natural home.

Definition 2.4.17 Let S be a non-empty subset of an inner product space V. We define $S^\perp$ to be the set of all vectors in V that are orthogonal to every vector in S i.e.

$$S^\perp := \{\mathbf{x} \in V : \langle \mathbf{x}, \mathbf{y} \rangle = 0 \text{ for all } \mathbf{y} \in S\}.$$

We call $S^\perp$ the **orthogonal complement of S**.

Example 2.4.18

(i) $\{\mathbf{0}_V\}^\perp = V$

(ii) $V^\perp = \{\mathbf{0}_V\}$

(iii) Let U be a subset of V. Then $U \cap U^\perp \subseteq \{\mathbf{0}_V\}$

▶

Proof.

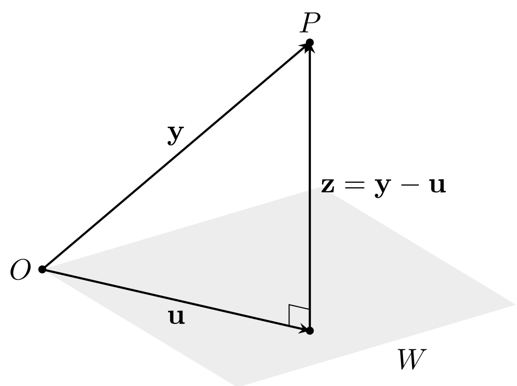
1. $\{\mathbf{0}_V\}^\perp \subseteq V$ is the collection of vectors in V that are orthogonal to $\mathbf{0}_V$. For the reverse inclusion, let $\mathbf{v} \in V$. Then $\langle \mathbf{v}, \mathbf{0}_V \rangle = 0 = \langle \mathbf{v}, 0\mathbf{0}_V \rangle = \overline{0}\langle \mathbf{v}, \mathbf{0}_V \rangle = 0$. Therefore, $\mathbf{v} \in \{\mathbf{0}_V\}^\perp$.

2. $V^\perp = \{\mathbf{w} \in V : \langle \mathbf{w}, \mathbf{v} \rangle \text{ for every } \mathbf{v} \in V\}$. Let $\mathbf{w} \in V^\perp$. We wish to show that $\mathbf{w} = \mathbf{0}_V$. Suppose for the sake of a contradiction that $\mathbf{w} \neq \mathbf{0}_V$. This is equivalent to $\langle \mathbf{w}, \mathbf{w} \rangle > 0$ which contradicts that $\mathbf{w} \in V^\perp$.

3. Let $\mathbf{u} \in U \cap U^\perp$. In particular, $\mathbf{u} \in U^\perp$ i.e. $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for any $\mathbf{v} \in V$. Take $\mathbf{v} = \mathbf{u}$ and so $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ i.e. $\mathbf{u} = \mathbf{0}_V$.

■

Now we return to the distance-minimisation problem. Let $\mathbf{y}$ be the position vector in the diagram below. The distance-minimisation problem takes the form of determining the vector $\mathbf{u} \in W$ that is "closest" to P in the sense that $\mathbf{z} := \mathbf{y} - \mathbf{u}$ has the shortest length.



Note that z is orthogonal to every vector in W, so $\mathbf{z} \in W^\perp$.

The following theorem offers a practical way of finding $\mathbf{u}$ in the case that W is a finite-dimensional subspace of an inner product space.

Theorem 2.4.19 (**6.6** [1, p. 350]) Let W be a finite-dimensional subspace of an inner product space V, and let $\mathbf{y} \in V$. Then there exist unique vectors $\mathbf{u} \in W$ and $\mathbf{z} \in W^{\perp}$ such that $\mathbf{y} = \mathbf{u} + \mathbf{z}$. Furthermore, if $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is an orthonormal basis for W, then

$$\mathbf{u} = \sum_{i=1}^{k} \langle \mathbf{y}, \mathbf{v}_i \rangle \mathbf{v}_i.$$

Proof. This is my own version of the proof.

Existence
Define $\mathbf{u}$ to be the linear combination of the components of $\mathbf{y}$ along the directions of the $\mathbf{v}_i$ in the orthonormal basis of W i.e.

$$\mathbf{u} = \sum_{i=1}^{k} \langle \mathbf{y}, \mathbf{v}_i \rangle \mathbf{v}_i \quad \text{is the part of } \mathbf{y} \text{ that lies in } W.$$

Then define $\mathbf{z} := \mathbf{y} - \mathbf{u}$ i.e. $\mathbf{y} = \mathbf{u} + \mathbf{z}$. Now we prove that $\mathbf{z} \in W^{\perp}$. For each $\mathbf{v}_j \in W$

$$\begin{aligned} \langle \mathbf{z}, \mathbf{v}_j \rangle &= \langle \mathbf{y}, \mathbf{v}_j \rangle - \sum_{i=1}^{k} \langle \mathbf{y}, \mathbf{v}_i \rangle \langle \mathbf{v}_i, \mathbf{v}_j \rangle \quad \text{by linearity} \\ &= \langle \mathbf{y}, \mathbf{v}_j \rangle - \langle \mathbf{y}, \mathbf{v}_j \rangle = 0. \end{aligned}$$

Since the $\mathbf{v}_1, \ldots, \mathbf{v}_k$ span W, this implies that $\mathbf{z} \in W^{\perp}$.

Uniqueness
Suppose that $\mathbf{y} = \mathbf{u} + \mathbf{z} = \mathbf{u}' + \mathbf{z}'$ where $\mathbf{u}' \in W$ and $\mathbf{z} \in W^{\perp}$. Then $\mathbf{u} - \mathbf{u}' = \mathbf{z} - \mathbf{z}' \in W \cap W^{\perp} = \{\mathbf{0}_V\}$ by **Examples 2.4.18 (iii)**. Thus, $\mathbf{u} - \mathbf{u}' = \mathbf{z} - \mathbf{z}' = \mathbf{0}_V$ so $\mathbf{u} = \mathbf{u}'$ and $\mathbf{z} = \mathbf{z}'$. ■

In fact, the above theorem is a direct sum decomposition of V into a finite-dimensional subspace W and its orthogonal complement $W^{\perp}$:

$$V = W \oplus W^{\perp}.$$

Remarks 2.4.20 In the uniqueness part of the proof of **Theorem 2.4.19**, we used that $W \cap W^{\perp} = \{\mathbf{0}_V\}$ (one of the conditions of the direct-sum-decomposition $V = W \oplus W^{\perp}$) implies the uniqueness of the vectors $\mathbf{u} \in W$ and $\mathbf{z} \in W^{\perp}$ in the decomposition $\mathbf{y} = \mathbf{u} + \mathbf{z}$. The reverse implication is also true, and so the conditions are equivalent
Proof. Suppose that every $\mathbf{y} \in V$ may be uniquely written as $\mathbf{y} = \mathbf{u} + \mathbf{z}$ with $\mathbf{u} \in W$ and $\mathbf{z} \in W^{\perp}$. Let $\mathbf{y} \in W \cap W^{\perp} \subseteq V$.

- Since $\mathbf{y} \in W$, we can write it as $\mathbf{y} = \mathbf{y} + \mathbf{0}_V$ where $\mathbf{0}_V \in W^{\perp}$.
- Since $\mathbf{y} \in W^{\perp}$, we can write it as $\mathbf{y} = \mathbf{0}_V + \mathbf{y}$ where $\mathbf{0}_V \in W$.

By the uniqueness of the decomposition, $\mathbf{y} + \mathbf{0}_V = \mathbf{0}_V + \mathbf{y}$ i.e. $\mathbf{y} = \mathbf{0}_V$ and $\mathbf{0}_V = \mathbf{y}$. Thus, $\mathbf{y} = \mathbf{0}_V$. ■

A very important corollary follows from **Theorem 2.4.19**.

Corollary 2.4.21 The vector $\mathbf{u}$ in the above discussion is the unique vector in W that is "closest" to $\mathbf{y}$ in the sense that

$$\mathbf{u} = \underset{\mathbf{w} \in W}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{w}\|.$$

Definition 2.4.22 We call $\mathbf{u}$ the **orthogonal projection of $\mathbf{y}$ on W**.

Proof. Let $\mathbf{y} = \mathbf{u} + \mathbf{z}$ as in the statement of the theorem. Let $\mathbf{w} \in W$. Then

$$\begin{aligned} \mathbf{y} - \mathbf{w} &= (\mathbf{u} + \mathbf{z}) - \mathbf{w} \\ &= \underbrace{(\mathbf{u} - \mathbf{w})}_{\in W} + \underbrace{\mathbf{z}}_{\in W^\perp} \end{aligned}$$

Therefore, $\mathbf{u} - \mathbf{w}$ and $\mathbf{z}$ are orthogonal, and by Pythagoras' theorem[3] we have that

$$\|\mathbf{y} - \mathbf{w}\|^2 = \|\mathbf{u} - \mathbf{w}\|^2 + \|\mathbf{z}\|^2$$

where $\|\mathbf{z}\|^2$ is a constant independent of $\mathbf{w}$, and so $\|\mathbf{y} - \mathbf{w}\|$ is minimised when $\|\mathbf{u} - \mathbf{w}\| = 0$ i.e. $\mathbf{u} = \mathbf{w}$. ■

Exercise 3 (13 [1, p. 355]) Let V be an inner product space, S and S_0 be subsets of V, and W be a finite-dimensional subspace of V. Prove the following.

(a) $S_0 \subseteq S \implies S^\perp \subseteq S_0^\perp$

(b) $S \subseteq (S^\perp)^\perp$

(c) $W = (W^\perp)^\perp$

Proof.

(a) Let $\mathbf{v} \in S_0$. We wish to show that if $\mathbf{w} \in S^\perp$ (i.e. that $\langle \mathbf{w}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in S$), then $\mathbf{w} \in S^\perp$, this means that $\langle \mathbf{w}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in S_0$. This should be clear since the former holds for S and so the latter holds for $S_0 \subseteq S$.

(b) Let $\mathbf{x} \in S$. We wish to show that $\mathbf{x} \in (S^\perp)^\perp$ i.e. we need to verify that $\langle \mathbf{x}, \mathbf{u} \rangle = 0$ for all $\mathbf{u} \in S^\perp$. Let $\mathbf{u} \in S^\perp$. This means that $\langle \mathbf{u}, \mathbf{s} \rangle = 0$ for all $\mathbf{s} \in S$. In particular, since $\mathbf{x} \in S$, we have that $\langle \mathbf{u}, \mathbf{s} \rangle = 0$. By conjugate symmetry, we have that

$$\langle \mathbf{x}, \mathbf{u} \rangle = \overline{\langle \mathbf{u}, \mathbf{x} \rangle} = \overline{0} = 0$$

and $\mathbf{u}$ was arbitrary in $S^\perp$ so $\mathbf{x} \in (S^\perp)^\perp$. Since $\operatorname{span}(S)$ is the smallest subspace of V containing S, and $(S^\perp)^\perp$ is a subspace of V, the claim follows.

(c) We first prove the following exercise:

Exercise 4 (6 [1, p. 354]) Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $\mathbf{x} \notin W$, prove that there exists a $\mathbf{y} \in V$ with $\mathbf{y} \in W^\perp$ but $\langle \mathbf{x}, \mathbf{y} \rangle \neq 0$.

Proof. By **Theorem 2.4.19**, there exist unique vectors $\mathbf{u} \in W$ and $\mathbf{z} \in W^\perp$ s.t. we have the unique decomposition $\mathbf{x} = \mathbf{u} + \mathbf{z}$. Since $\mathbf{x} \notin W$, we must have $\mathbf{z} \neq \mathbf{0}_V$ (for if $\mathbf{z} = \mathbf{0}_V$, then $\mathbf{x} = \mathbf{u} \in W$). Take $\mathbf{y}$ to be this unique $\mathbf{z}$. Then

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= \langle \mathbf{u} + \mathbf{z}, \mathbf{z} \rangle \\ &= \langle \mathbf{u}, \mathbf{z} \rangle + \langle \mathbf{z}, \mathbf{z} \rangle \\ &= 0 + \|\mathbf{z}\|^2 \end{aligned}$$

■

Part (b) already demonstrates that $S \subseteq (S^\perp)^\perp$ for any subset. In particular, we may take S to be equal to the subspace W of V. For the reverse inclusion, let $\mathbf{x} \in (W^\perp)^\perp$. We wish to show that $\mathbf{x} \in W$. Suppose for a contradiction that $\mathbf{x} \notin W$. By **Exercise 4**, there exists some $\mathbf{y} \in W^\perp$ s.t. $\langle \mathbf{x}, \mathbf{y} \rangle \neq 0$. Since $\mathbf{x} \in (W^\perp)^\perp$, $\langle \mathbf{x}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in W^\perp$. Take $\mathbf{v} = \mathbf{y}$. Then $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ which is a contradiction.

■

[3] Which states that if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors, then $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

Theorem 2.4.23 (**6.7** [1, p. 352]) Suppose that $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthonormal subset of an n-dimensional inner product space V. Then

(a) S can be extended to an orthonormal basis $(\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n)$ for V.

(b) If we denote by $W := \operatorname{span}(S)$, then the added vectors $S_1 := (\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n)$ form an orthonormal basis for $W^\perp$.

(c) If W is any subspace of V, then $\dim(V) = \dim(W) + \dim(W^\perp)$.

Proof.

(a) Extending the orthonormal set to an orthonormal basis

Since $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthonormal (and hence orthogonal) set in an inner product space, **Corollary 2.4.14** tells us that S is linearly independent. We can extend this collection to an ordered basis $S' = \{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_n\}$ of V. Apply the Gram-Schmidt process to orthogonalise S'. By Exercise 8 [1, p. 354],

Exercise 5 If $\mathbf{w}_1, \ldots, \mathbf{w}_n$ is an orthogonal set of non-zero vectors, then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ obtained from the $\mathbf{w}_j$ by the Gram-Schmidt process satisfy $\mathbf{v}_i = \mathbf{w}_i$ for $i = 1, \ldots, n$.

the first k vectors of S' remain unchanged. After normalising the last $n - k$ vectors ($\mathbf{v}_j := \mathbf{w}_j / \|\mathbf{w}_j\|$ for $j = k+1, \ldots, n$), we have an orthonormal basis

$$(\underbrace{\mathbf{v}_1, \ldots, \mathbf{v}_k}_{S}, \underbrace{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n}_{S_1}) \text{ for } V.$$

(b) If $W = \operatorname{span}(S)$, then the added vectors form an orthonormal basis for $W^\perp$

S_1 is linearly independent as a subset of the basis above. Since $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for $i \leqslant k$ and $j > k$, we have that $S_1 \subseteq W^\perp$. What remains to be seen is that S_1 spans $W^\perp$, from which we conclude (with its orthonormality) that S_1 is a basis for $W^\perp$. For any $\mathbf{x} \in V$, we have its representation with respect to the orthonormal basis for V

$$\mathbf{x} = \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i.$$

If $\mathbf{x} \in W^\perp$, then $\langle \mathbf{x}, \mathbf{v}_i \rangle = 0$ for $j = 1, \ldots, k$ and so

$$\mathbf{x} = \sum_{i=k+1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i \in \operatorname{span}(S_1).$$

(c) Dimension formula

Let W be any subspace of V. Since V is a finite-dimensional inner product space, so too is W, and so W admits an orthonormal basis (by the Gram-Schmidt process). By (a) and (b), we can extend this orthonormal basis of W to an orthonormal basis

$$(\underbrace{\mathbf{v}_1, \ldots, \mathbf{v}_k}_{S}, \underbrace{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n}_{S_1}) \text{ for } V$$

with $W = \operatorname{span}(S)$ and $W^\perp = \operatorname{span}(S_1)$. Therefore, $\dim(W) = k$, $\dim(W^\perp)$, and

$$\dim(V) = n = k + (n - k) = \dim(W) + \dim(W^\perp).$$

■

2.5 Duality

2.5.1 Dual Space

Definition 2.5.1 Given a vector space V over a field $\mathbb{K}$, we call $V^* := \mathrm{Hom}_{\mathbb{K}}(V; \mathbb{K})$ the **(algebraic) dual space of V**.

Elements of V^* i.e. linear maps $f : V \to \mathbb{K}$ called **linear functionals**. Let $E = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ be a basis for V. What are the basic elements of V^*?

A choice of basis for V can be equivalently viewed as a linear isomorphism $\Phi_E : \mathbb{K}^{n,1} \to V$ defined uniquely by its action on the standard basis $(\mathfrak{e}_1, \dots, \mathfrak{e}_n)$ of $\mathbb{K}^{n,1}$. We capture the above objects in a single diagram:

$$\begin{tikzcd}
V \arrow[dr, "f\in V^*"] & \\
& \mathbb{K} \\
\mathbb{K}^{n,1} \arrow[uu, "\Phi_E"] \arrow[ur, dashed, "?"'] &
\end{tikzcd}$$

We understand linear maps $\mathbb{K}^{n,1} \to \mathbb{K}$ very well. They are uniquely defined by their action on a basis, so in our case we may consider any $\psi : \mathbb{K}^{n,1} \to \mathbb{K}$ to be of the form

$$\psi\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \sum_{i=1}^{n} a_i x_i$$

where $a_i := \psi(\mathfrak{e}_i)$. It's easily demonstrated that the basic elements of $(\mathbb{K}^{n,1})^*$ are the coordinate projections

$$\begin{aligned} \pi_i : \mathbb{K}^{n,1} &\to \mathbb{K} \\ : \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} &\longmapsto x_i \end{aligned}$$

Now, we may use our basis Φ_E (a bijection), to pass from discussing the basic linear functionals (the coordinate projections) that linearly combine to any $\psi : \mathbb{K}^{n,1} \to \mathbb{K}$ (i.e. $\psi \in (\mathbb{K}^{n,1})^*$), to discuss the basic elements of V^* that combine to form any $f \in V^*$. We do this by defining

$$\mathbf{e}^i := \pi_i \circ \Phi_E^{-1} : V \to \mathbb{K}.$$

By **Proposition 1.2.4 (iii)**, the linear isomorphism Φ_E sends the ordered basis $(\pi_1, \dots, \pi_n)$ of $(\mathbb{K}^{n,1})^*$ to the ordered basis $(\mathbf{e}^1, \dots, \mathbf{e}^n)$ of V^*. This also demonstrates that if V is finite-dimensional, then $\dim(V) = \dim(V^*)$.

Each basis element satisfies

$$\mathbf{e}^i(\mathbf{e}_j) := (\pi_i \circ \Phi_E^{-1})(\mathbf{e}_j) = \pi_i(\mathfrak{e}_j) = \delta_{ij},$$

and by linearity we have for any $\mathbf{x} = \alpha_1 \mathbf{e}_1 + \ldots + \alpha_n \mathbf{e}_n$ that

$$\mathbf{e}^i(\mathbf{x}) = \mathbf{e}^i\left(\sum_{j=1}^{n} \alpha_j \mathbf{e}_j\right) = \sum_{j=1}^{n} \alpha_j \mathbf{e}^i(\mathbf{e}_j) = \sum_{j=1}^{n} \alpha_j \delta_{ij} = \alpha_i$$

so our dual basis map $\mathbf{e}^i$ extracts the i^{th} coordinate α_i from $\mathbf{x}$.

2.5.2 Dual Map

Let V and W be finite-dimensional vector spaces over $\mathbb{K}$. For any linear map $T\colon V \to W$, and a linear functional $g\colon W \to \mathbb{K}$, we may define a new linear functional $g \circ T$ by pre-composition with T:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, dashed, "T \circ g"'] & W \arrow[d, "g"] \\ & \mathbb{K} \end{tikzcd}$$

This gives us a map T^*, called the **dual map**, between[4] the dual spaces

$$\begin{aligned} T^*\colon W^* &\to V^* \\ : g &\longmapsto g \circ T \end{aligned}$$

2.5.3 Transpose

Now we wish to find the matrix representation of T^* with respect to the dual bases. Explicitly, let $E = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ and $F = (\mathbf{f}_1, \dots, \mathbf{f}_m)$ be bases for V and W respectively. Also, let $E^* = (\mathbf{e}^1, \dots, \mathbf{e}^n)$ and $F^* = (\mathbf{f}^1, \dots, \mathbf{f}^n)$ be the dual bases of V^* and W^* respectively. The j^{th} column of $A = [F, T, E]$ is the coordinates of $T(\mathbf{e}_j)$ w.r.t F:

$$[F, T(\mathbf{e}_j)] = \begin{bmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{mj} \end{bmatrix}$$

In similar spirit, we wish to compute the coordinates of $T^*(\mathbf{f}_k)$ with respect to E^* in order to determine the k^{th} column of the matrix representation $[E^*, T^*, F^*]$ of T^*. Since $T^*(\mathbf{f}^k) \in V^*$, it's of the form

$$T^*(\mathbf{f}^k) = \sum_{j=1}^{n} c_j \mathbf{e}^j$$

and we determine c_j by evaluating $T^*(\mathbf{f}^k)$ at $\mathbf{e}_j$:

$$c_l = (T^*(\mathbf{f}^k))(\mathbf{e}_l) = \mathbf{f}^k(T(\mathbf{e}_l)) = \mathbf{f}^k\left(\sum_{i=1}^{m} \alpha_{il}\mathbf{f}_i\right) = \sum_{i=1}^{m} \alpha_{il}\mathbf{f}^k(\mathbf{f}_i) = \sum_{i=1}^{m} \alpha_{il}\delta_{ki} = \alpha_{kl}$$

The k^{th} column of $[E^*, T^*, F^*]$ is

$$[E^*, T^*] = \begin{bmatrix} \alpha_{k1} \\ \alpha_{k2} \\ \vdots \\ \alpha_{kn} \end{bmatrix}$$

and therefore the matrix representation is

$$[E^*, T^*, F^*] = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{m1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{mn} \end{bmatrix} = A^\top = [F, T, E]^\top$$

[4]True to the naming convention, this map reverses the order of $T\colon V \to W$ to a map $W^* \to V^*$.

2.6 Inner Product Gives Canonical Isomorphism

2.6.1 Flat

By fixing a basis for a finite-dimensional vector space, we obtain a basis-dependent isomorphism $\mathbb{K}^{n,1} \to V$. For a *finite-dimensional* inner product space V, we obtain an isomorphism between V and its algebraic dual $V^* := \mathrm{Hom}_{\mathbb{K}}(V;\mathbb{K})$. The existence of this isomorphism does not depend on a choice of basis and so we call it a canonical isomorphism.

- One direction of the isomorphism is the **flat map**[5]

$$\begin{aligned}\flat\colon V &\to V^* \\ : \mathbf{v} &\longmapsto \langle\,\cdot\,,\mathbf{v}\rangle\end{aligned}$$

Lemma 2.6.1 $\flat\colon V \to V^*$ is a linear isomorphism.

Proof. For linearity, let $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $\alpha, \beta \in \mathbb{K}$. Then for any $\mathbf{w} \in V$,

$$\begin{aligned}\flat(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2)(\mathbf{w}) &:= \langle\alpha\mathbf{v}_1 + \beta\mathbf{v}_2, \mathbf{w}\rangle \\ &= \alpha\langle\mathbf{v}_1,\mathbf{w}\rangle + \beta\langle\mathbf{v}_2,\mathbf{w}\rangle \quad \text{by linearity} \\ &=: \alpha\flat(\mathbf{v}_1)(\mathbf{w}) + \beta\flat(\mathbf{v}_2)(\mathbf{w}).\end{aligned}$$

For injectivity, we show the equivalent condition that $\flat$ has trivial kernel. Namely, suppose that $\flat(\mathbf{v}) = 0_{\mathrm{Hom}}$. Then $0_{\mathbb{K}} = \flat(\mathbf{v})(\mathbf{v}) := \langle\mathbf{v},\mathbf{v}\rangle \iff \mathbf{v} = \mathbf{0}_V$ so $\flat$ has trivial kernel. This means that $\flat$ is injective. Since V and V^* are finite-dimensional with the same dimension, injectivity of $\flat$ is equivalent to, by the Rank-Nullity Theorem, $\flat$ being surjective. Thus, $\flat$ is bijective and so it has an inverse $\sharp\colon V^* \to V$. By **Proposition 1.2.3**, $\flat$ is a linear bijection i.e. a linear isomorphism, and we get that $\sharp$ is a linear isomorphism for free. ■

Definition 2.6.2 The canonical isomorphism $\flat$ and its inverse $\sharp$ are called **musical isomorphisms**.

2.6.2 Sharp (Riesz representation theorem)

What is the functional form of $\sharp\colon V^* \to V$? It's a map that takes as input any linear functional $g \in V^*$ and outputs the unique (recall that this is a bijection so we have uniqueness) $\mathbf{y} \in V$ with $g = \langle\mathbf{x},\mathbf{y}\rangle$ for all $\mathbf{x} \in V$.

Definition 2.6.3 The **sharp map** $\sharp\colon V^* \to V$ is defined by for any $g \in V^*$ by the equation

$$\langle\mathbf{x}, \sharp(g)\rangle = g(\mathbf{x}) \quad \text{for all } \mathbf{x} \in V.$$

This above definition is in fact the content of the Riesz representation theorem:

Theorem 2.6.4 (Riesz Representation Theorem) Let V be a finite-dimensional inner product space over $\mathbb{K}$, and let $g\colon V \to \mathbb{K}$ be a linear transformation. Then there exists a unique vector $\mathbf{y} \in V$ s.t. $g(\mathbf{x}) = \langle\mathbf{x},\mathbf{y}\rangle$ for all $\mathbf{x} \in V$.

Proof 1. Existence: Let $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ be an orthonormal basis for V, and $\mathbf{x} \in V$. Then we may write any $\mathbf{x} \in V$ uniquely as

$$\mathbf{x} = \sum_{j=1}^{n}\langle\mathbf{x},\mathbf{v}_j\rangle\mathbf{v}_j.$$

[5] An easy way to remember this is that we wish for the output map $\flat(\mathbf{v})$ to be linear, and $\langle\,\cdot\,,\,\cdot\,\rangle$ is defined to be linear in its first entry, so we fix the second entry.

Suppose that g is non-zero.[6] Apply g.

$$g(\mathbf{x}) = \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{v}_j \rangle \overbrace{g(\mathbf{v}_j)}^{\in \mathbb{K}} = \sum_{j=1}^{n} \left\langle \mathbf{x}, \overline{g(\mathbf{v}_j)} \mathbf{v}_j \right\rangle = \left\langle \mathbf{x}, \sum_{j=1}^{n} \overline{g(\mathbf{v}_j)} \mathbf{v}_j \right\rangle$$

<u>Uniqueness</u>: Suppose that there's another $\mathbf{y}' \in V$ s.t. $g(\mathbf{x}) = \langle \mathbf{x}, \mathbf{y}' \rangle$ for all $\mathbf{x} \in V$. Then

$$g(\mathbf{x}) = \langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y}' \rangle \quad \text{for all } \mathbf{x} \in V$$

and so $\mathbf{y} = \mathbf{y}'$. ■

Proof 2. Suppose that g is non-zero i.e. there exists some $\mathbf{x} \in V$ s.t. $g(\mathbf{x}) \neq 0$. The only subspaces of $\mathbb{K}$ are $\{0_{\mathbb{K}}\}$ and $\mathbb{K}$ itself, and $\text{image}(g) \supsetneq \{0_{\mathbb{K}}\}$ so $\text{image}(g) = \mathbb{K}$. Since $\ker(g) \subsetneq V$ is a linear subspace, and V is finite-dimensional, then $\ker(g)$ is finite-dimensional. We may express $V = \ker(g) \oplus (\ker(g))^{\perp}$ so every $\mathbf{x}$ can be uniquely expressed as $\mathbf{x} = \mathbf{u} + \mathbf{z}$ with $\mathbf{u} \in \ker(g)$ and $\mathbf{z} \in (\ker(g))^{\perp}$, and the following dimension formula holds

$$\dim(V) = \dim(\ker g) + \dim((\ker g)^{\perp}).$$

By the rank-nullity theorem applied to g,

$$\begin{aligned} \dim(V) &= \dim(\ker g) + \dim(\text{image}\, g) \\ &= \dim(\ker g) + 1 \end{aligned}$$

i.e. $\ker g$ has codimension 1 in V. It follows by combining the dimension equations that $(\ker g)^{\perp}$ is 1-dimensional. Let $\mathbf{w}$ be a unit vector spanning $(\ker g)^{\perp}$ i.e. every element of $(\ker g)^{\perp}$ is of the form $\alpha \mathbf{w}$ where $\alpha \in \mathbb{K}$. Thus, our decomposition becomes

$$\mathbf{x} = \mathbf{u} + \alpha \mathbf{w}.$$

Now we make the following two observations:

- $g(\mathbf{x}) = g(\mathbf{u} + \alpha \mathbf{w}) = g(\mathbf{u}) + \alpha g(\mathbf{w}) = 0_{\mathbb{K}} + \alpha g(\mathbf{w}) = \alpha g(\mathbf{w})$
- $\langle \mathbf{x}, \mathbf{w} \rangle = \langle \mathbf{u} + \alpha \mathbf{w}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \alpha \langle \mathbf{w}, \mathbf{w} \rangle = 0_{\mathbb{K}} + \alpha = \alpha$

Combine the two to obtain

$$g(\mathbf{x}) = \langle \mathbf{x}, \mathbf{w} \rangle g(\mathbf{w}) = \left\langle \mathbf{x}, \overline{g(\mathbf{w})} \mathbf{w} \right\rangle$$

and let $\mathbf{y} = \overline{g(\mathbf{w})} \mathbf{w}$ to conclude the proof. ■

Both proofs end up with different forms of the same unique vector $\mathbf{y}$. I will denote them by

$$\mathbf{y}_1 = \sum_{j=1}^{n} \overline{g(\mathbf{v}_j)} \mathbf{v}_j \quad \text{and} \quad \mathbf{y}_2 = \overline{g(\mathbf{w})} \mathbf{w}, \text{ respectively.}$$

In the 2nd proof, we can take $\{\mathbf{w}\}$ which is a basis for $(\ker g)^{\perp}$, and extend it by **Theorem 2.4.23 (b)** to an orthonormal basis

$$(\underbrace{\mathbf{v}_1 = \mathbf{w}}_{S}, \underbrace{\mathbf{v}_2, \ldots, \mathbf{v}_n}_{S_1}) \text{ for } V$$

with $(\ker g)^{\perp} = \text{span}(S)$ and $\ker g = \text{span}(S_1)$. Thus, $g(\mathbf{v}_i) = 0$ for $i = 2, \ldots, n$. Therefore,

$$\mathbf{y}_1 = \overline{g(\mathbf{v}_1)} \mathbf{v}_1 = \overline{g(\mathbf{w})} \mathbf{w} = \mathbf{y}_2.$$

[6] The zero case is trivial by letting $\mathbf{y} = \mathbf{0}_V$ so that $g(\mathbf{x}) = \langle \mathbf{x}, \mathbf{y} \rangle = 0$ for all $\mathbf{x} \in V$.

2.6.3 Adjoint

Let V and W be finite-dimensional inner product spaces, and $T^*\colon W^* \to V^*$ be the dual map defined by $T^*(g) = g \circ T$. In this case, we can use the musical isomorphisms we've defined in order to bring T^* back to W and V.

$$\begin{tikzcd} W^* \arrow[r, "T^*"] & V^* \arrow[d, "\sharp_V"] \\ W \arrow[u, "\flat_W"] \arrow[r, "?"'] & V \end{tikzcd}$$

The composite map at the bottom of this diagram is called the **adjoint of T**, denoted by $T^\dagger\colon W \to V$, and defined by

$$T^\dagger = \sharp_V \circ T^* \circ \flat_W.$$

For any $\mathbf{w} \in W$

$$\begin{aligned} T^\dagger(\mathbf{w}) &:= (\sharp_V \circ T^* \circ \flat_W)(\mathbf{w}) \\ &= \sharp_V(T^*(\flat_W(\mathbf{w}))) \\ &:= \sharp_V(\flat_W(\mathbf{w}) \circ T) \end{aligned}$$

Now note that $T^* \circ \flat_W(\mathbf{w}) \in V^*$, and for any $\mathbf{v} \in V$ it takes the form

$$(\flat_W(\mathbf{w}) \circ T)(\mathbf{v}) = \langle T(\mathbf{v}), \mathbf{w} \rangle_W$$

Recall that $\sharp_V\colon V^* \to V$ takes as input $g \in V^*$ and outputs the unique $\mathbf{y} \in V$ for which

$$g(\mathbf{v}) = \langle \mathbf{v}, \mathbf{y} \rangle_V \qquad \text{for all } \mathbf{v} \in V.$$

Let $g = T^* \circ \flat_W(\mathbf{w})$ so $\mathbf{y} := T^\dagger(\mathbf{w})$, and so we have for all $\mathbf{v} \in V$ and $\mathbf{w} \in W$ that

$$\begin{aligned} \langle T(\mathbf{v}), \mathbf{w} \rangle_W &= (T^* \circ \flat_W(\mathbf{w}))(\mathbf{v}) = g(\mathbf{v}) \\ &= \langle \mathbf{v}, \sharp_V(g) \rangle_V = \langle \mathbf{v}, \sharp_V(T^* \circ \flat_W(\mathbf{w})) \rangle_V = \langle \mathbf{v}, T^\dagger(\mathbf{w}) \rangle_V. \end{aligned}$$

We state this property as a corollary:

Corollary 2.6.5 (**Adjoint Property**) The adjoint $T^\dagger$ of T satisfies

$$\langle T(\mathbf{v}), \mathbf{w} \rangle_W = \langle \mathbf{v}, T^\dagger(\mathbf{w}) \rangle_V$$

for all $\mathbf{v} \in V$ and $\mathbf{w} \in W$.

Right now, my intuition for the adjoint comes from the adjoint property above. In a sense, it says that measuring $T(\mathbf{v})$ against $\mathbf{w}$ in W with $\langle\,\cdot\,,\,\cdot\,\rangle_W$ is "the same" as measuring $\mathbf{v}$ against $T^\dagger(\mathbf{w})$ in V with $\langle\,\cdot\,,\,\cdot\,\rangle_V$ in V.

If we restrict this perspective to a linear operator $T\colon V \to V$, then this reveals more of a "mirror image". Namely, that measuring $T(\mathbf{v}_1)$ against $\mathbf{v}_2$ is the same as measuring $\mathbf{v}_1$ against $T^\dagger(\mathbf{v}_2)$.

The adjoint $T^\dagger$ (through this property) helps to explain how **types** of linear maps interact with the inner products on their domain and codomain.[a] The method of interaction is how we classify linear maps $T\colon V \to W$ into **types** e.g. a linear operator $T\colon V \to V$ is called self-adjoint if $T^\dagger = T$.

[a]Looking back, this is correct intuition. More on this in **Chapter 3**.

2.6.4 Conjugate Transpose

Now we find the matrix representation of $T^\dagger\colon W \to V$ with respect to the orthonormal bases $F = (\mathbf{f}_1, \ldots, \mathbf{f}_m)$ of W and $E = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ of V. Write

$$T(\mathbf{e}_j) = \sum_{i=1}^{m} a_{ij}\mathbf{f}_i \quad \text{and} \quad T^\dagger(\mathbf{f}_k) = \sum_{i=1}^{n} b_{ij}\mathbf{e}_i.$$

Then by the adjoint property for inner products, we have

$$\langle T(\mathbf{e}_j), \mathbf{f}_k\rangle_W = \langle \mathbf{e}_j, T^\dagger(\mathbf{f}_k)\rangle_V,$$

the LHS and RHS of which are

$$\begin{aligned} \langle T(\mathbf{e}_j), \mathbf{f}_k\rangle_W &= \left\langle \sum_{i=1}^{m} a_{ij}\mathbf{f}_i, \mathbf{f}_k \right\rangle \\ &= \sum_{i=1}^{m} a_{ij}\langle \mathbf{f}_i, \mathbf{f}_k\rangle \\ &= a_{kj} \end{aligned} \qquad \begin{aligned} \langle \mathbf{e}_j, T^\dagger(\mathbf{f}_k)\rangle_W &= \left\langle \mathbf{e}_j, \sum_{i=1}^{n} b_{ik}\mathbf{e}_i \right\rangle \\ &= \sum_{i=1}^{n} \overline{b_{ik}}\langle \mathbf{e}_j, \mathbf{e}_i\rangle \\ &= \overline{b_{jk}} \end{aligned}$$

Finally, we conclude that $b_{jk} = \overline{a_{kj}}$ which is the statement that

$$[E, T^\dagger, F] = \overline{A}^\top = \overline{[F, T, E]}^\top.$$

Now that we have the matrix representation of the adjoint with respect to the aforementioned bases, we may speak of the coordinate form of the adjoint property. Recall that the adjoint property states

$$\langle T(\mathbf{v}), \mathbf{w}\rangle_W = \langle \mathbf{v}, T^\dagger(\mathbf{w})\rangle_V \quad \text{for all } \mathbf{v} \in V,\ \mathbf{w} \in W.$$

By **Example 2.4.10**, we may re-write the LHS and RHS as

$$\begin{aligned} \langle T(\mathbf{v}), \mathbf{w}\rangle_W &= \langle [F, T(\mathbf{v})], [F, \mathbf{w}]\rangle_{\mathbb{K}^{m,1}} \\ &= \langle A[E, \mathbf{v}], [F, \mathbf{w}]\rangle_{\mathbb{K}^{m,1}} \end{aligned} \qquad \begin{aligned} \langle \mathbf{v}, T^\dagger(\mathbf{w})\rangle_W &= \langle [E, \mathbf{v}], \overline{A}^\top [F, \mathbf{w}]\rangle_{\mathbb{K}^{n,1}} \\ &= \langle [E, \mathbf{v}], \overline{A}^\top [F, \mathbf{w}]\rangle_{\mathbb{K}^{n,1}} \end{aligned}$$

We state this property as a corollary:

Corollary 2.6.6 (**Adjoint Property in Coordinates**) Let $A = [F, T, E]$ be the matrix corresponding to the linear map $T\colon V \to W$, where E is a basis for V, and F is a basis for W. Then A satisfies

$$\langle A[E, \mathbf{v}], [F, \mathbf{w}]\rangle_{\mathbb{K}^{m,1}} = \langle [E, \mathbf{v}], \overline{A}^\top [F, \mathbf{w}]\rangle_{\mathbb{K}^{n,1}}$$

for all $\mathbf{v} \in V$ and $\mathbf{w} \in W$.

Remarks 2.6.7 Authors often give names to $[E, \mathbf{v}]$ and $[F, \mathbf{w}]$ like $\mathbf{x}$ and $\mathbf{y}$, respectively, in which case the defining property looks less cluttered:

$$\langle A\mathbf{x}, \mathbf{y}\rangle_{\mathbb{K}^{m,1}} = \langle \mathbf{x}, \overline{A}^\top \mathbf{y}\rangle \quad \text{for all } \mathbf{x} \in \mathbb{K}^{n,1},\ \mathbf{y} \in \mathbb{K}^{m,1}.$$

Lemma 2.6.8 (The Algebra of Adjoints) Let V be a finite-dimensional[7] inner product space, and $T, U\colon V \to V$ be linear operators.

(a) $(T+U)^\dagger = T^\dagger + U^\dagger$

(b) $(\alpha T)^\dagger = \overline{\alpha} T^\dagger$

(c) $(T \circ U)^\dagger = U^\dagger \circ T^\dagger$

(d) $(T^\dagger)^\dagger = T$

(We often denote this double adjoint by $T^{\dagger\dagger}$.)

(e) $(\mathrm{id}_V)^\dagger = \mathrm{id}_V$

Proof. Let $\mathbf{x}, \mathbf{y} \in V$, and $\alpha \in \mathbb{K}$.

(a) Note that $T+U$ is a linear operator on the finite-dimensional inner product space V, and so its adjoint $(T+U)^\dagger$ exists and is the unique linear operator that satisfies $\langle (T+U)\mathbf{x}, \mathbf{y}\rangle = \left\langle \mathbf{x}, (T+U)^\dagger \mathbf{y}\right\rangle$ for every $\mathbf{x}, \mathbf{y} \in V$. Now note that

$$\begin{aligned}
\langle \mathbf{x}, (T+U)^\dagger(\mathbf{y})\rangle &= \langle (T+U)(\mathbf{x}), \mathbf{y}\rangle \\
&= \langle T(\mathbf{x}), \mathbf{y}\rangle + \langle U(\mathbf{x}), \mathbf{y}\rangle \\
&= \langle \mathbf{x}, T^\dagger(\mathbf{y})\rangle + \langle \mathbf{x}, U^\dagger(\mathbf{y})\rangle \quad \text{by the adjoint properties of } T \text{ and } U \\
&= \langle \mathbf{x}, (T^\dagger + U^\dagger)(\mathbf{y})\rangle.
\end{aligned}$$

Therefore, $(T+U)^\dagger = T^\dagger + U^\dagger$ as maps.

(d) For every $\mathbf{x}, \mathbf{y} \in V$,

$$\begin{aligned}
\langle \mathbf{x}, (T^\dagger)^\dagger(\mathbf{y})\rangle &= \langle T^\dagger(\mathbf{x}), \mathbf{y}\rangle \quad \text{by the adjoint property of } T^\dagger \\
&= \langle \mathbf{x}, T(\mathbf{y})\rangle \quad \text{by the adjoint property of } T
\end{aligned}$$

i.e. $\langle \mathbf{x}, ((T^\dagger)^\dagger - T)(\mathbf{y})\rangle = 0$ for all $\mathbf{x}, \mathbf{y} \in V$. By **Lemma 2.4.3**, this implies that $((T^\dagger)^\dagger - T)(\mathbf{y}) = 0$ for every $\mathbf{y} \in V$ i.e. $(T^\dagger)^\dagger = T$ as maps.

■

Remarks 2.6.9 As a small terminological remark, the dual map $T^*\colon W^* \to V^*$ defined earlier is also called the **Banach adjoint**. The 'Banach' in the name is because the construction doesn't depend on an inner product, and is more generally defined for Banach vector spaces V and W. The 'adjoint' part of the name is described by the following logic:

T^* is defined for any $g \in W^*$ by

$$T^*(g) := g(T),$$

and for any $\mathbf{v} \in V$:

$$(T^* g)(\mathbf{v}) = g(T(\mathbf{v})).$$

Adopting some non-standard notation, we can write that expression in a way that pairs the map and the input by

$$[\![T^* g, \mathbf{v}]\!] = [\![g, T(\mathbf{v})]\!]$$

which looks very similar to the adjoint property of $T^\dagger$:

$$\langle T^\dagger(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{x}, T(\mathbf{y})\rangle \quad \text{for all } \mathbf{x}, \mathbf{y} \in V.$$

[7]The finite-dimensionality guarantees that the adjoints $T^\dagger$ and $U^\dagger$ exist, but this theorem extends to infinite-dimensional inner product spaces if we assume the existence of the appropriate adjoints.

Chapter 3

Interaction with Adjoint, and Preservation

This chapter has one overarching goal — to define several types of linear maps (or operators where appropriate) between inner product spaces by how they interact with their own adjoint (assuming it exists). This can be summarised into the following tree:

$$\begin{tikzcd}
 & & \text{Self-adjoint} \arrow[d, Rightarrow] \arrow[r, dash] & \text{Diagonalisable with real eigenvalues} \\
\text{How do } T^\dagger \text{ and } T \text{ interact?} \arrow[urr, "T^\dagger = T"] \arrow[rr, "T^\dagger \circ T = T \circ T^\dagger"] \arrow[drr, "T^\dagger \circ T = \mathrm{id}_V"'] & & \text{Normal} \arrow[r, dash] & \text{The Complex Spectral Theorem} \\
 & & \text{Unitary} \arrow[u, Rightarrow] \arrow[r, dash] & \text{Diagonalisable over } \mathbb{C} \text{ with eigenvals of modulus 1}
\end{tikzcd}$$

Since every self-adjoint operator is normal, and every unitary operator is normal, the complex spectral theorem for normal operators implies the diagonalisability of the other two types of operator.

The chapter roadmap is split into 3 parts:

- Normal operator route:

  Look for conditions under which there exists an orthonormal basis for V consisting of eigenvectors of a linear operator $T\colon V \to V$ on a finite-dimensional inner product space. (This is tantamount to looking for a diagonalisability criterion on an inner product space.)

  - Prove Schur's Theorem which asserts the conditions under which a linear operator on a finite-dimensional inner product space admits an upper triangular representation.
  - This leads to the definition of a normal linear operator on an inner product space.

- Self-adjoint operator route:

  For Euclidean spaces, replace the normality condition by self-adjointness in order to guarantee the existence of such a basis (diagonalisation).

- Unitary (orthogonal) operator route:

  Discuss metric isometries, linear isometries between normed vector spaces, and characterise linear isometries between inner product spaces. This offers the definition of a unitary (orthogonal) operator.

  - Use the complex and real spectral theorems to characterise the diagonalisability of the matrices corresponding to normal and self-adjoint operators, respectively.

3.1 Normal Operator Route

3.1.1 Schur's Theorem

Theorem 3.1.1 (Schur) Let T be a linear operator on a finite-dimensional inner product space V. Suppose that the characteristic polynomial of T splits. Then there exists an orthonormal basis

E of V s.t. $[E, T, E]$ is an upper triangular matrix.

Proof Sketch. The method of proof is by induction on the dimension of V. For $n = 1$ the result is clear. We begin by observing that because the characteristic polynomial of T splits, it has an eigenvalue and a corresponding eigenvector.

The high-level goal of the proof is to **find a subspace** $W \subseteq V$ of dimension 1 for which its orthogonal complement $W^\perp$ in V is T-invariant. If we can achieve this, then V admits the direct sum decomposition $V = W^\perp \oplus W$. If we can demonstrate that the linear operator $T|_W$ splits, then we can apply the inductive hypothesis (that if the characteristic polynomial of $T\colon V \to V$ splits, then there exists an orthonormal basis E of V s.t. $[E, T, E]$ is upper triangular) to $T|_{W^\perp}\colon W^\perp \to W^\perp$ so there exists an orthonormal basis E' of $W^\perp$ with respect to which $[E', T|_{W^\perp}, E']$ is upper triangular. From this, we can extend E' to an orthonormal basis E for V (by **Theorem 2.4.23**) and hopefully conclude that $[E, T, E]$ is a block upper triangular matrix. ◧

How we actually go about **this** needs some unpacking. The subspace $W^\perp$ of co-dimension 1 in V is a hyperplane. Every hyperplane of V is the kernel of a non-zero functional on V as demonstrated via the Rank-Nullity Theorem in the 2nd proof of the Riesz representation theorem.

What does it mean for a hyperplane $H = \ker(\varphi)$ to be T-invariant? Suppose that $H = \ker(\varphi)$ is T-invariant. This means that if $\mathbf{h} \in H$, then $T(\mathbf{h}) \in H$ i.e. $\varphi(\mathbf{h}) = 0 \implies \varphi(T(\mathbf{h})) = 0$ (and notice that this may be re-written as $(T^* \circ \varphi)(\mathbf{h}) = 0$). This means that $H = \ker(\varphi) \subseteq \ker(\varphi \circ T)$. The reverse inclusion also holds by a dimension[1] argument: for non-zero $\varphi \circ T$, $\dim(\ker(\varphi \circ T)) = \dim(V) - 1 = \dim(\ker(\varphi))$ so $\ker(\varphi) = \ker(\varphi \circ T)$. Now consider the following lemma.

Lemma 3.1.2 Any two linear functionals that vanish on the same hyperplane are scalar multiples of one another.

Proof. Suppose that ϕ and ψ are non-zero linear functions on V with $\ker(\phi) = \ker(\psi) = H$. Since H has co-dimension 1, choose some vector $\mathbf{v}_0 \in V \setminus H$. Then $\phi(\mathbf{v}_0) \neq 0$ and $\psi(\mathbf{v}_0) \neq 0$. Take any vector $\mathbf{v} \in V$. We may uniquely decompose it as $\mathbf{v} = \mathbf{h} + \alpha\mathbf{v}_0$ with $\mathbf{h} \in H$, $\alpha \in \mathbb{K}$. Then $\phi(\mathbf{v}) = \phi(\mathbf{h} + \alpha\mathbf{v}_0) = \phi(\mathbf{h}) + \alpha\phi(\mathbf{v}_0) = \alpha\phi(\mathbf{v}_0)$. Similarly, $\psi(\mathbf{v}) = \alpha\psi(\mathbf{v}_0)$. Combining the two gives

$$\psi(\mathbf{v}) = \frac{\psi(\mathbf{v}_0)}{\phi(\mathbf{v}_0)}\phi(\mathbf{v}) =: \lambda\phi(\mathbf{v})$$

as desired. ■

Since we've demonstrated that $\ker(\varphi \circ T) = \ker(\varphi)$, the lemma tells us that $T^*(\varphi) = \varphi \circ T = \lambda\varphi$ i.e. that $\varphi \in V^*$ is an eigenvector of $T^*\colon V^* \to V^*$. In summary, we've shown that for $0_{V^*} \neq \varphi \in V^*$, $\ker(\varphi)$ being T-invariant implies that φ is an eigenvector of T^*. It would be wonderful if the converse also held so we had a characterisation of a hyperplane being T-invariant.

Suppose that φ is an eigenvector of T^* i.e. that there exists some scalar $\lambda \in \mathbb{K}$ s.t. $T^*(\varphi) := \varphi \circ T = \lambda\varphi$. We wish to show that $H = \ker(\varphi)$ is T-invariant. Take $\mathbf{v} \in H = \ker(\varphi)$ i.e. $\varphi(\mathbf{v}) = 0$. Then

$$\begin{aligned}(T^*\varphi)(\mathbf{v}) &:= (\varphi \circ T)(\mathbf{v}) \\ &= \lambda\varphi(\mathbf{v}) \\ &= 0\end{aligned}$$

Therefore, $T(\mathbf{v}) \in \ker(\varphi) = H$ i.e. H is T-invariant. We package this into a single lemma:

Lemma 3.1.3 Let $\varphi \in V^*$. The subspace $\ker(\varphi) \subseteq V$ is T-invariant $\iff$ φ is an eigenvector of $T^*\colon V^* \to V^*$.

Recall that the adjoint $T^\dagger\colon V \to V$ is the map defined by

$$T^\dagger := \sharp \circ T^* \circ \flat,$$

[1]Explicitly, any two vector subspaces of a finite-dimensional vector space that share the same dimension are isomorphic.

satisfying the adjoint property

$$\langle T(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{x}, T^\dagger(\mathbf{y})\rangle \quad \text{for all } \mathbf{x}, \mathbf{y} \in V.$$

Suppose that $\varphi \in V^*$ is an eigenvector of T^* i.e. there exists some $\lambda \in \mathbb{K}$ s.t. $T^*(\varphi) = \lambda\varphi$. By the Riesz representation theorem, there exists a unique $\mathbf{z} \in V$ s.t. $\varphi = \flat(\mathbf{z}) = \langle\,\cdot\,, \mathbf{z}\rangle$. Thus, we may re-write the eigenvector criterion as

$$T^*(\flat(\mathbf{z})) = \flat(\mathbf{z}) \circ T = \lambda\flat(\mathbf{z})$$

i.e. for every $\mathbf{v} \in V$

$$\langle T(\mathbf{v}), \mathbf{z}\rangle = (\flat(z) \circ T)(\mathbf{v}) = (\lambda\flat(\mathbf{z}))(\mathbf{v}) = \lambda\langle \mathbf{v}, \mathbf{z}\rangle.$$

By the adjoint property, and conjugate linearity in the 2^{nd} entry of $\flat(\mathbf{z})$, we obtain

$$\langle \mathbf{v}, T^\dagger(\mathbf{z})\rangle = \left\langle \mathbf{v}, \overline{\lambda}\mathbf{z}\right\rangle \quad \text{for all } \mathbf{v} \in V$$

and by **Lemma 2.4.3**, $T^\dagger(\mathbf{z}) = \overline{\lambda}\mathbf{z}$ i.e. $\mathbf{z}$ is an eigenvector of $T^\dagger$. The converse also holds readily, so $\flat$ gives a one-to-one correspondence between eigenvectors φ of T^*, and eigenvectors $\mathbf{z}$ of $T^\dagger$.

So we've found a nice correspondence here. Let $\varphi \in V^*$. Then

$$\ker(\varphi) \text{ is } T\text{-invariant} \iff T^*(\varphi) = \lambda\varphi \iff T^\dagger(\mathbf{z}) = \overline{\lambda}\mathbf{z}.$$

Now we're in a position to prove the theorem.

Proof Proper. The method of proof is by induction on the dimension of V. We begin by assuming that the characteristic polynomial of T splits. Thus, T has an eigenvalue λ and a corresponding eigenvector $\mathbf{z}$. If we can show that $\mathbf{z}$ is an eigenvector of $T^\dagger$, then we can apply all the preceding theory.

Lemma 3.1.4 ([1, p. 369]) Let T be a linear operator on a finite-dimensional inner product space V. If T has an eigenvector, then so does $T^\dagger$.

Proof. Suppose that $\mathbf{z} \neq \mathbf{0}_V$ is an eigenvector of T corresponding to the eigenvalue λ. For any $\mathbf{v} \in V$:

$$\begin{aligned} 0 = \langle \mathbf{0}_V, \mathbf{v}\rangle &= \langle (T - \lambda\,\mathrm{id}_V)(\mathbf{z}), \mathbf{v}\rangle \\ &= \langle \mathbf{z}, (T - \lambda\,\mathrm{id}_V)^\dagger(\mathbf{v})\rangle \\ &= \langle \mathbf{z}, (T^\dagger - \overline{\lambda}\,\mathrm{id}_V)(\mathbf{v})\rangle \end{aligned}$$

so $\mathbf{z}$ is in the orthogonal complement of image$(T^\dagger - \overline{\lambda}\,\mathrm{id}_V)$. Since V is finite-dimensional, so is the subspace image$(T^\dagger - \overline{\lambda}\,\mathrm{id}_V)$ and so V admits the direct sum decomposition

$$V = \mathrm{image}(T^\dagger - \overline{\lambda}\,\mathrm{id}_V) \oplus \underbrace{(\mathrm{image}(T^\dagger - \overline{\lambda}\,\mathrm{id}_V))^\perp}_{\substack{\supsetneq\{\mathbf{0}_V\} \\ \text{since it contains } \mathbf{z}}}.$$

Thus, $(T^\dagger - \overline{\lambda}\,\mathrm{id}_V)$ is not[2] surjective, and equivalently not injective, which is equivalent to $(T^\dagger - \overline{\lambda}\,\mathrm{id}_V)$ having a non-zero kernel. Any vector in this kernel is an eigenvector of $T^\dagger$ with corresponding eigenvalue $\overline{\lambda}$. ■

Thus, $\mathbf{z}$ is an eigenvalue of the adjoint $T^\dagger$ with corresponding eigenvalue $\overline{\lambda}$. We may assume without loss of generality that $\mathbf{z}$ is a unit eigenvector. Now consider $W = \mathrm{span}(\{\mathbf{z}\})$. Is $W^\perp$ a T-invariant subspace of V? Take $\mathbf{y} \in W^\perp$ i.e. $\langle \mathbf{y}, \mathbf{z}\rangle = 0$. We wish to show that $T(\mathbf{y}) \in W^\perp$.

$$\begin{aligned} \langle T(\mathbf{y}), \mathbf{z}\rangle &= \langle \mathbf{y}, T^\dagger(\mathbf{z})\rangle \\ &= \left\langle \mathbf{y}, \overline{\lambda}\mathbf{z}\right\rangle \\ &= 0 \quad \text{since } \mathbf{y} \in W^\perp. \end{aligned}$$

[2] This wasn't immediately clear to me but this made it click: Suppose that some linear operator $T\colon V \to V$ is onto. Then image$(T) = V$, and $(\mathrm{image}(T))^\perp = V^\perp = \{\mathbf{0}_V\}$. But we have that $(\mathrm{image}(T))^\perp \supsetneq \{\mathbf{0}_V\}$ and so the claim follows.

Note that $W^\perp$ has co-dimension 1 in V (as $V = W^\perp \oplus W$ where $\dim(W) = \dim(\operatorname{span}(\{\mathbf{z}\})) = 1$). Thus, $W^\perp$ is a hyperplane that is T-invariant. By the correspondence we established earlier, we connect $\mathbf{z}$ being an eigenvector of $T^\dagger$ corresponding to the eigenvalue $\overline{\lambda}$ with $W^\perp$ being the kernel of some $\varphi \in V^*$ which is an eigenvector of T^* with eigenvalue λ. Now we can follow the steps outlined in the proof sketch.

Theorem 3.1.5 (**5.21** [1, p. 314]) Let $T\colon V \to V$ be a linear operator on a finite-dimensional inner product space V. If W is a T-invariant subspace of V, then the characteristic polynomial of $T|_W$ divides the characteristic polynomial of T.

Since $W^\perp$ is a subspace of V, and the characteristic polynomial of T splits, then so too does the characteristic polynomial of $T|_{W^\perp}$. This means that there exists an orthonormal basis E' of $W^\perp$ with respect to which $[E', T|_{W^\perp}, E']$ is an upper-triangular matrix. We can extend E' to an orthonormal basis E (by appending $\mathbf{z}$ to E') for V by **Theorem 2.4.23** where

$$\begin{aligned} V &= W^\perp \oplus W \\ &= W^\perp \oplus \operatorname{span}(\{\mathbf{z}\}). \end{aligned}$$

Since $W^\perp$ is T-invariant, every basis element for $W^\perp$ has final coordinate zero with respect to the basis E. The coordinate of $\mathbf{z}$ (the basis element for W) with respect to E has a λ as its final entry, and there's no restriction on the other entries. Thus, the matrix representation of T with respect to E is

$$[E, T, E] = \begin{bmatrix} [E', T|_{W^\perp}, E'] & * \\ \mathbf{0}_{1,\dim(V)-1} & \lambda \end{bmatrix}$$

which is upper triangular. ∎

If V admits an orthonormal basis of eigenvectors of T, then $[E, T, E]$ is diagonal and so

$$[E, T^\dagger, E] = \overline{[E, T, E]}^\top$$

is also diagonal. Since diagonal matrices commute, so too do T and $T^\perp$. We give a name to operators that exhibit this behaviour.

Definition 3.1.6

- Let V be an inner product space, and $T\colon V \to V$ be a linear operator on V. If $T \circ T^\dagger = T^\dagger \circ T$, then we call T **normal**.
- If a matrix $A \in \mathbb{K}^{n,n}$ satisfies $A(\overline{A}^\top) = \overline{A}^\top A$, then we call A **normal**.

Remarks 3.1.7 Note that T is normal if $T^\dagger \circ T = T \circ T^\dagger$, and by the fundamental isomorphism between linear maps and matrices this is equivalent to

$$[E, T^\dagger, E][E, T, E] = [E, T, E][E, T^\dagger, E].$$

Since E is an orthonormal basis for V, it follows that the matrix representation of the adjoint $T^\dagger$ is the conjugate transpose $[E, T^\dagger, E] = \overline{[E, T, E]}^\top$, our equality reduces to

$$\overline{[E, T, E]}^\top [E, T, E] = [E, T, E]\overline{[E, T, E]}^\top$$

which is precisely the condition that $[E, T, E]$ is a normal matrix. i.e.

Lemma 3.1.8 Let E be an orthonormal basis of V. Then $T\colon V \to V$ is normal iff $[E, T, E]$ is normal.

3.1.2 Some Facts About Normal Operators

This subsection is dedicated to facts about normal operators that will be used in subsequent proofs.

Lemma 3.1.9 (Properties of Normal Operators) Let T be a normal operator on an inner product space V.

(a) T-normal and its adjoint are isometric i.e. for every $\mathbf{x} \in V$, $\|T(\mathbf{x})\| = \left\|T^\dagger(\mathbf{x})\right\|$.

(b) $T + \alpha I$ is normal for every $\alpha \in \mathbb{K}$.

(c) If $\mathbf{x}$ is an eigenvector of T corresponding to the eigenvalue λ, then $\mathbf{x}$ is also an eigenvector of $T^\dagger$ but with corresponding eigenvalue $\overline{\lambda}$.

(d) Let λ_1, λ_2 be distinct eigenvalues of T. Their corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2 \in V$ are orthogonal.

Remarks 3.1.10

- Part (a) says that a normal operator and its adjoint are isometric i.e. they scale any vector by the same amount.
- Part (b) is a useful theorem when discussing eigenvalues. Namely, if we know that T is normal, then $T - \lambda \operatorname{id}_V$ is also normal, and we may apply the facts we know about normal operators to $T - \lambda \operatorname{id}_V$.
- Part (d) is innocuously important. The theorem we're building up to (**Theorem 3.1.12**) will tell us that we can find an orthonormal basis whose directions (after rotating the standard basis into them), scales according to the eigenvalues.

Proof.

(a) Let $\mathbf{x} \in V$. Then

$$\begin{aligned}
\|T(\mathbf{x})\|^2 &:= \langle T(\mathbf{x}), T(\mathbf{x})\rangle \\
&= \langle T^\dagger(T(\mathbf{x})), \mathbf{x}\rangle && \text{by the adjoint property} \\
&= \langle T(T^\dagger(\mathbf{x})), \mathbf{x}\rangle && \text{by normality} \\
&= \langle T^\dagger(\mathbf{x}), T^\dagger(\mathbf{x})\rangle && \text{by the adjoint property} \\
&=: \|T^\dagger(\mathbf{x})\|^2
\end{aligned}$$

(b) This part follows very easily by expanding $(T+\alpha \operatorname{id}_V)$ post and pre-composed with its adjoint and verifying equality. I will do so in a single chain of equalities for brevity:

$$\begin{aligned}
&(T + \alpha \operatorname{id}_V)^\dagger \circ (T + \alpha \operatorname{id}_V) \\
&= (T^\dagger + \overline{\alpha} \operatorname{id}_V) \circ (T + \alpha \operatorname{id}_V) \\
&= (T^\dagger + \overline{\alpha} \operatorname{id}_V) \circ (T + \alpha \operatorname{id}_V) \\
&= (T^\dagger \circ T) + \alpha(T^\dagger \circ \operatorname{id}_V) + \overline{\alpha}(\operatorname{id}_V \circ T) + \overline{\alpha}\alpha(\operatorname{id}_V \circ \operatorname{id}_V) \\
&= (T \circ T^\dagger) + \alpha(\operatorname{id}_V \circ T^\dagger) + \overline{\alpha}(T \circ \operatorname{id}_V) + \overline{\alpha}\alpha(\operatorname{id}_V \circ \operatorname{id}_V) && \text{by normality and } \operatorname{id}_V \text{ commutes} \\
&= (T \circ T^\dagger) + \overline{\alpha}(T \circ \operatorname{id}_V) + \alpha(\operatorname{id}_V \circ T^\dagger) + \overline{\alpha}\alpha(\operatorname{id}_V \circ \operatorname{id}_V) \\
&= (T + \alpha \operatorname{id}_V) \circ (T + \alpha \operatorname{id}_V)^\dagger
\end{aligned}$$

(c) Suppose that $T(\mathbf{x}) = \lambda \mathbf{x}$ for some $\mathbf{x} \in V \setminus \{\mathbf{0}_V\}$, and $\lambda \in \mathbb{K}$. Then consider $U = T - \lambda \operatorname{id}_V$ so that $U(\mathbf{x}) = 0$. By **Lemma 3.1.9 (ii)**, $U = T - \lambda \operatorname{id}_V$ is also normal.

$$\begin{aligned}
0 = \|U(\mathbf{x})\| &= \|U^\dagger(\mathbf{x})\| && \text{since } U \text{ and } U^\dagger \text{ are isometric} \\
&= \|(T^\dagger - \overline{\lambda} \operatorname{id}_V)(\mathbf{x})\|
\end{aligned}$$

and by separation of points, $T^\dagger(\mathbf{x}) = \overline{\lambda}\mathbf{x}$.

(d) First inkling was to consider $\langle T(\mathbf{v}_1), T(\mathbf{v}_2)\rangle$ which can be manipulated into both

$$\lambda_1\overline{\lambda_2}\langle \mathbf{v}_1, \mathbf{v}_2\rangle = \langle T(\mathbf{v}_1), T(\mathbf{v}_2)\rangle = \lambda_2\overline{\lambda_2}\langle \mathbf{v}_1, \mathbf{v}_2\rangle.$$

and from here I noticed that there's an extra $\overline{\lambda_2}$ term that could've been done away with had I instead started with $\langle T(\mathbf{v}_1), \mathbf{v}_2\rangle$. So consider:

$$\begin{aligned}\langle T(\mathbf{v}_1), \mathbf{v}_2\rangle &= \langle \lambda_1\mathbf{v}_1, \mathbf{v}_2\rangle \\ &= \lambda_1\langle \mathbf{v}_1, \mathbf{v}_2\rangle\end{aligned} \qquad \begin{aligned}\langle T(\mathbf{v}_1), \mathbf{v}_2\rangle &= \langle \mathbf{v}_1, T^{\dagger}(\mathbf{v}_2)\rangle \\ &= \left\langle \mathbf{v}_1, \overline{\lambda_2}\mathbf{v}_2\right\rangle \\ &= \lambda_2\langle \mathbf{v}_1, \mathbf{v}_2\rangle\end{aligned}$$

i.e. $(\lambda_1 - \lambda_2)\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0$ and because $\lambda_1 \neq \lambda_2$, this forces $\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0$ i.e. that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.

■

3.1.3 Complex Spectral Theorem

It's easy to cook up an example of a normal linear operator T on a real inner product space V for which there doesn't exist an orthonormal basis for V consisting of eigenvectors of T.

Example 3.1.11 ([1, p. 371]) Consider the linear operator on $\mathbb{R}^2$ representing an anti-clockwise rotation by $\theta \in (0, \pi)$. $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is uniquely defined by its action on the standard basis $E = (\mathfrak{e}_1, \mathfrak{e}_2)$ of $\mathbb{R}^2$:

$$T(\mathfrak{e}_1) = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix} = \cos(\theta)\mathfrak{e}_1 + \sin(\theta)\mathfrak{e}_2.$$

$$T(\mathfrak{e}_2) = T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix} = -\sin(\theta)\mathfrak{e}_1 + \cos(\theta)\mathfrak{e}_2.$$

Thus, the matrix representation of T w.r.t E is

$$[E, T, E] = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}.$$

The eigenvalues of T are the roots of the characteristic equation

$$0 = \det([E, T, E] - xI_2) = \det\left(\begin{bmatrix}\cos(\theta) - x & -\sin(\theta)\\ \sin(\theta) & \cos(\theta) - x\end{bmatrix}\right) = x^2 - 2x\cos(\theta) + 1.$$

The eigenvalues are $\lambda_{1,2} = \cos(\theta) \pm i\sin(\theta)$ and for $\theta \in (0, \pi)$, we observe that $\lambda_{1,2} \in \mathbb{C} \setminus \mathbb{R}$ i.e. T doesn't have any **real** eigenvalues. Thus, normality is insufficient for the existence of an orthonormal basis for V consisting of eigenvectors of T. ▶

However, normality does suffice if V is a complex inner product space.

Theorem 3.1.12 (**Complex Spectral Theorem for Normal Operators**) Let T be a linear operator on a finite-dimensional complex inner product space V. Then T is normal iff there exists an orthonormal basis for V consisting of eigenvectors of T.

Proof.

$\Longrightarrow$ Suppose that T is normal. Since we're in a complex inner product space, the characteristic polynomial of T splits (by the fundamental theorem of algebra). By Schur's theorem, we obtain an orthonormal basis $E = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ for V s.t. $A = [E, T, E]$ is upper triangular. Since A is upper triangular, $\mathbf{v}_1$ is certainly an eigenvector of T.

We'll demonstrate that all the $\mathbf{v}_i$ are eigenvalues of T by the principle of strong[3] induction. Assume that $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$ are all eigenvectors of T, where for each $j < k$, $\mathbf{v}_j$ corresponds to the eigenvalue λ_j of T. By **Lemma 3.1.9 (iii)**, $\mathbf{v}_j$ is an eigenvector of $T^\dagger$ with corresponding eigenvalue $\overline{\lambda_j}$.

Recall that for a linear operator on a finite dimensional inner product space with orthonormal basis $E = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$, every $\mathbf{x} \in V$ may be uniquely represented as a linear combination of the basis vectors

$$\mathbf{x} = \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i.$$

Since E is an orthonormal basis for V, we may write $T(\mathbf{v}_k)$ uniquely as

$$\begin{aligned}
T(\mathbf{v}_k) &= \sum_{i=1}^{n} \langle T(\mathbf{v}_k), \mathbf{v}_i \rangle \mathbf{v}_i \\
&= \sum_{i=1}^{k} \langle T(\mathbf{v}_k), \mathbf{v}_i \rangle \mathbf{v}_i \qquad \text{since } [E, T, E] \text{ is upper triangular} \\
&= \langle T(\mathbf{v}_k), \mathbf{v}_1 \rangle \mathbf{v}_1 + \ldots + \langle T(\mathbf{v}_k), \mathbf{v}_{k-1} \rangle \mathbf{v}_{k-1} + \langle T(\mathbf{v}_k), \mathbf{v}_k \rangle \mathbf{v}_k \\
&= \langle \mathbf{v}_k, T^\dagger(\mathbf{v}_1) \rangle \mathbf{v}_1 + \ldots + \langle \mathbf{v}_k, T^\dagger(\mathbf{v}_{k-1}) \rangle \mathbf{v}_{k-1} + \langle T(\mathbf{v}_k), \mathbf{v}_k \rangle \mathbf{v}_k \\
&= \left\langle \mathbf{v}_k, \overline{\lambda_1} \mathbf{v}_1 \right\rangle \mathbf{v}_1 + \ldots + \left\langle \mathbf{v}_k, \overline{\lambda_{k-1}} \mathbf{v}_{k-1} \right\rangle \mathbf{v}_j + \langle T(\mathbf{v}_k), \mathbf{v}_k \rangle \mathbf{v}_k \\
&= 0 + \ldots + 0 + \langle T(\mathbf{v}_k), \mathbf{v}_k \rangle \mathbf{v}_k \qquad \text{by orthogonality of } E \\
&= \lambda_k \mathbf{v}_k \qquad \text{where we've defined } \lambda_k := \langle T(\mathbf{v}_k), \mathbf{v}_k \rangle.
\end{aligned}$$

Therefore, $\mathbf{v}_k$ is an eigenvector of T. It follows that all the vectors in E are eigenvectors of T.

$\Longleftarrow$ For the converse, let $E = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an orthonormal basis for V consisting of eigenvectors of T.

$\Longrightarrow$ $[E, T, E]$ is diagonal

$\Longrightarrow$ $[E, T^\dagger, E] = \overline{[E, T, E]}^\top$ is also diagonal, and diagonal matrices commute

$\Longrightarrow$ $[E, T, E]$ is normal

$\Longleftrightarrow$ T is normal.

■

[3] Proving that a statement $P(n)$ holds under the assumption that all the statements $P(m)$ hold for $N_0 \ni m < n$.

3.2 Self-Adjoint Operator Route

For real inner product spaces, we've already seen an example where normality is insufficient for diagonalisability. We replace normality by the stronger condition that $T = T^\dagger$ in order to guarantee the existence of an orthonormal basis of V consisting of eigenvectors of T.

Definition 3.2.1

- Let $T: V \to V$ be a linear operator on an inner product space V. We say that T is **self-adjoint** (or Hermitian) if $T = T^\dagger$.
- A complex matrix $A \in \mathbb{C}^{n,n}$ is called **self-adjoint** (or **Hermitian**) if $A = \overline{A}^\top$.

Remarks 3.2.2

- If E is an orthonormal basis for V, then T is self-adjoint iff $[E, T, E]$ is self-adjoint.
- If $A \in \mathbb{C}^{n,n}$ has only real entries, then self-adjoint means real symmetric.
- If V is a *real* inner product space, and E is an orthonormal basis for V, then T is a self-adjoint operator on V iff $[E, T, E]$ is self-adjoint i.e. a *real symmetric matrix*.

Without some motivation, I find it difficult to mentally invest in a lemma, lemma, proposition, theorem, lemma, theorem, blah... setup. To this end, I'll summarise the significance of some concepts:

- Eigenvectors of a normal operator corresponding to distinct eigenvalues of said operator, are orthogonal vectors. (Already proven)
- Self-adjoint operators are normal. This is self-evident.
- The eigenvalues (if they exist) of any self-adjoint operator over any (real or complex) inner product space are **real**.
- We will encounter a spectral theorem that says any self-adjoint operator can be diagonalised.
- The eigenvalues are real so if we take a basis of eigenvectors (which represent the axes of a coordinate system), then self-adjoint operators act on these vectors along these axes by pure scaling.
    - Had these eigenvalues been complex, then the action of the corresponding operator would involve some rotation too.

Lemma 3.2.3 Let $T: V \to V$ be a self-adjoint operator on a finite-dimensional inner product space. Then:

(a) Every eigenvalue of T is real.

(b) Suppose that V is a real inner product space. Then the characteristic polynomial of T splits.

Proof.

(a) Let $\mathbf{x} \neq \mathbf{0}_V$ be an eigenvector of T i.e. $T(\mathbf{x}) = \lambda\mathbf{x}$. Every self-adjoint operator is normal. Thus, $\mathbf{x}$ is an eigenvector of $T^\dagger$ with eigenvalue $\overline{\lambda}$. Therefore,

$$\begin{aligned} \lambda\mathbf{x} = T(\mathbf{x}) &= T^\dagger(\mathbf{x}) \quad \text{since } T \text{ is self-adjoint} \\ &= \overline{\lambda}\mathbf{x} \end{aligned}$$

Thus, $\lambda = \overline{\lambda}$ i.e. $\lambda \in \mathbb{R}$.

(b) Let $\dim(V) = n$, and E be an orthonormal basis for V. Denote by $A := [E, T, E]$. The characteristic polynomial of T is $\det(A - xI_n)$. Since T is self-adjoint, and E is an orthonormal basis for V, then A is self-adjoint i.e. A is real symmetric ($A = A^\top$).

Now we may consider A as a complex matrix acting on $\mathbb{C}^{n,1}$ (equipped with the standard inner product) i.e. the corresponding linear map $L_A \colon \mathbb{C}^{n,1} \to \mathbb{C}^{n,1}$ is defined by $\mathbf{x} \longmapsto A\mathbf{x}$. Let γ denote the standard ordered (orthonormal) basis of $\mathbb{C}^{n,1}$. Thus, $A = [\gamma, L_A, \gamma]$ and observe that A is Hermitian so L_A is self-adjoint.

By the fundamental theorem of algebra, the characteristic polynomial of L_A splits over $\mathbb{C}$. Therefore, L_A has at least one complex eigenvalue $\lambda \in \mathbb{C}$ with corresponding eigenvector $\mathbf{v} \in \mathbb{C}^{n,1}$. By part (a), every eigenvalue of L_A is real, and so the characteristic polynomial of L_A splits over $\mathbb{R}$. Both L_A and T have the same characteristic polynomial, thus concluding the proof.

■

Now we may present the real spectral theorem for self-adjoint operators.

Theorem 3.2.4 (**Real Spectral Theorem for Self-Adjoint Operators**) Let T be a linear operator on a finite-dimensional real inner product space V. Then T is self-adjoint iff there exists an orthonormal basis E for V consisting of eigenvectors of T.

Proof.

$\Longrightarrow$ Suppose that T is self-adjoint. By the previous lemma, its characteristic polynomial splits, and by Schur's theorem there exists an orthonormal basis E of V with respect to which $A = [E, T, E]$ is upper triangular. Since T is self-adjoint, A is symmetric, and every symmetric upper-triangular matrix is diagonal. Thus, E is a basis for V consisting of eigenvectors of T.

$\Longleftarrow$ Suppose that E is an orthonormal basis for V consisting of eigenvectors of T. Thus, $A = [E, T, E]$ is a diagonal matrix with real entries. Since E is an orthonormal basis, we have the identity $[E, T^\dagger, E] = \overline{[E, T, E]}^\top$ and so it follows that

$$[E, T^\dagger, E] = \overline{[E, T, E]}^\top = \overline{A}^\top = A$$

i.e. $[E, T, E]$ is self-adjoint which is equivalent to T being self-adjoint.

■

3.3 Unitary (Orthogonal) Operator Route

3.3.1 Isometry Bonanza

This section is dedicated to building up some terminology to discuss orthogonal matrices (which correspond to a type of map that preserves the inner product).

We've already seen that any inner product induces a norm, and hence defines a normed vector space that supports the notion of "length". In addition, any norm on a vector space can be used to define a metric (a distance function) $d\colon V \times V \to \mathbb{R}$ on said vector space by $d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|$.

In general, a metric can be defined on any non-empty set X. The following definition reflects this and we'll work down from the most general case of metric spaces all the way to inner product spaces.

Definition 3.3.1 Let X be some non-empty set. A **metric on X** is a map $d\colon X \times X \to \mathbb{R}$ that satisfies the following properties:

- Non-negativity: For all $x, y \in X$, $d(x, y) \geqslant 0$.
- Separation of points: $d(x, y) = 0 \iff x = y$.
- Symmetry: For any $x, y \in X$, we have that $d(x, y) = d(y, x)$.
- Triangle inequality: For any $x, y, z \in X$, $d(x, y) \leqslant d(x, z) + d(z, y)$.

The pair (X, d) is called a **metric space**.

Now we have all the machinery required to discuss the types of maps that arise by investigating how distance, length, and inner products are preserved.

Definition 3.3.2 Let X and Y be non-empty sets. A map $T\colon (X, d_X) \to (Y, d_Y)$ between metric spaces is called a **metric isometry** if for all $x, y \in X$:

$$d_Y(T(x), T(y)) = d_X(x, y).$$

Definition 3.3.3 Suppose that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed vector spaces. A map $T\colon X \to Y$ is called a **linear isometry** if T is linear and for all $\mathbf{x} \in X$:

$$\|T(\mathbf{x})\|_Y = \|\mathbf{x}\|_X.$$

Remarks 3.3.4

- In the definition of a linear isometry, the norm-preservation condition is equivalent to the metric condition. Let d_X and d_Y be metrics on X and Y induced by their respective norms. Then
$$\begin{aligned} d_Y(T(\mathbf{x}), T(\mathbf{y})) &:= \|T(\mathbf{x}) - T(\mathbf{y})\|_Y \\ &= \|T(\mathbf{x} - \mathbf{y})\|_Y \\ &= \|\mathbf{x} - \mathbf{y}\|_X \\ &=: d_X(\mathbf{x}, \mathbf{y}) \end{aligned}$$
- A linear isometry is also called an isometric embedding.
- Metric isometries are necessarily injective. Indeed, suppose that $T(x) = T(y)$. Then $0 = d_Y(T(x), T(y)) = d_X(x, y)$ i.e. $x = y$.
- A linear isometry between normed vector spaces is necessarily injective. The proof is very simple. Indeed, if $T(\mathbf{x}) = T(\mathbf{y})$ then
$$\begin{aligned} 0 = \|T(\mathbf{x}) - T(\mathbf{y})\|_Y = \|T(\mathbf{x} - \mathbf{y})\|_Y &\quad \text{by linearity} \\ = \|\mathbf{x} - \mathbf{y}\|_X &\quad \text{since } T \text{ is a linear isometry} \end{aligned}$$
i.e. $\mathbf{x} = \mathbf{y}$.

- If a linear isometry is also surjective, then we call it an isometric isomorphism.

Now suppose that $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ are inner product spaces, and let $\|\cdot\|_V$ and $\|\cdot\|_W$ be the norms induced by the respective inner products e.g. $\|\cdot\|_V := \sqrt{\langle\cdot,\cdot\rangle_V}$.

Lemma 3.3.5 A linear map $T\colon V \to W$ is a linear isometry (i.e. preserves norms) iff for all $\mathbf{x}, \mathbf{y} \in V$:

$$\langle T(\mathbf{x}), T(\mathbf{y})\rangle_W = \langle \mathbf{x}, \mathbf{y}\rangle_V.$$

$\blacksquare$[4]

Lemma 3.3.6 Now suppose that $T\colon V \to W$ admits a unique adjoint map $T^\dagger\colon W \to V$. Then T is a linear isometry iff $T^\dagger \circ T = \mathrm{id}_V$.

Proof.

$\implies$ Suppose that T is a linear isometry i.e. inner-product-preserving

$$\langle \mathbf{x}, \mathbf{y}\rangle = \langle T(\mathbf{x}), T(\mathbf{y})\rangle \quad \text{for all } \mathbf{x}, \mathbf{y} \in V.$$

From the adjoint property, it follows that for all $\mathbf{x}, \mathbf{y} \in V$:

$$\langle \mathbf{x}, \mathbf{y}\rangle = \langle T(\mathbf{x}), T(\mathbf{y})\rangle = \langle \mathbf{x}, T^\dagger(T(\mathbf{y}))\rangle$$

and by **Lemma 2.4.3**, $(\mathrm{id}_V - (T^\dagger \circ T))(\mathbf{y}) = \mathbf{0}_V$ for all $\mathbf{y} \in V$. Thus, $\mathrm{id}_V = T^\dagger \circ T$ as maps.

$\impliedby$ We wish to show that T is norm-preserving. For any $\mathbf{x} \in V$:

$$\begin{aligned}
\|\mathbf{x}\|_V^2 &:= \langle \mathbf{x}, \mathbf{x}\rangle_V \\
&= \langle \mathbf{x}, (T^\dagger \circ T)(\mathbf{x})\rangle_V \\
&= \langle T(\mathbf{x}), T(\mathbf{x})\rangle_W \\
&=: \|T(\mathbf{x})\|_W^2.
\end{aligned}$$

$\blacksquare$

The above lemma tells us that the admission of a left-inverse defines a linear isometry. The admission of a left-inverse is equivalent to the map being injective.[5]

If we suppose further that a linear isometry is surjective, then it's invertible, and we obtain the notion of an isomorphism between inner product spaces:

Definition 3.3.7

- A bijective linear isometry $T\colon V \to W$ between complex inner product spaces V and W is called **unitary**.
- A bijective linear isometry $T\colon V \to W$ between Euclidean spaces V and W is called **orthogonal**.

Note that T is unitary iff $\underbrace{T^\dagger \circ T = T \circ T^\dagger = \mathrm{id}_V}_{\text{i.e. } T^{-1} = T^\dagger}$.

Definition 3.3.8 Equivalently, by **Lemma 3.3.5** we may define:

[4] The proof follows readily from the polarisation identity.
[5] We already knew this from **Remarks 3.3.4** but it's nice to have some consistency.

- A bijective linear map $T: V \to W$ between complex inner product spaces V and W is called **unitary** if it preserves inner products i.e. for all $\mathbf{x}, \mathbf{y} \in V$:

$$\langle T(\mathbf{x}), T(\mathbf{y}) \rangle_W = \langle \mathbf{x}, \mathbf{y} \rangle_V.$$

- If V and W are Euclidean spaces, then we say T is **orthogonal**.

From the definitions above, we can quite immediately make a comment on any eigenvalue of a unitary (or orthogonal) operator[6] T. Indeed, if $T: V \to V$ is a unitary operator on V, then it preserves the inner product on V, and has unique adjoint $T^\dagger = T^{-1}$. Suppose that λ is an eigenvector of T with corresponding eigenvector $\mathbf{v} \in V \setminus \{\mathbf{0}_V\}$ i.e. $T(\mathbf{v}) = \lambda \mathbf{v}$. Then

$$\begin{aligned} \|\mathbf{v}\|^2 &:= \langle \mathbf{v}, \mathbf{v} \rangle \\ &= \langle T(\mathbf{v}), T(\mathbf{v}) \rangle \\ &=: \|T(\mathbf{v})\|^2 \\ &= \|\lambda \mathbf{v}\|^2 \\ &= |\lambda|^2 \|\mathbf{v}\|^2. \end{aligned}$$

Since $\mathbf{v} \neq \mathbf{0}_V$, $\|\mathbf{v}\| \neq 0$, and so we conclude that every eigenvalue of a unitary operator has absolute value 1. The same can be said for orthogonal operators.

3.3.2 Matrix representations

On the level of matrix representations, let's work generally with inner product spaces V and W over $\mathbb{K}$. Let $E = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ and $F = (\mathbf{f}_1, \ldots, \mathbf{f}_m)$ be the respective orthonormal bases of V and W. Let $T: V \to W$ be a linear map. Then $A = [F, T, E]$ is the matrix representation of T with respect to the bases E and F, and the j^{th} column of A is the coordinates of $T(\mathbf{e}_j)$ with respect to F. Namely, since F is a basis for W we may uniquely write

$$T(\mathbf{e}_j) = \sum_{i=1}^{m} a_{ij} \mathbf{f}_i$$

for some scalars $\alpha_{1j}, \ldots, \alpha_{mj} \in \mathbb{C}$. Then we can extract the a_{ij} entry by taking the inner product

$$\begin{aligned} \langle T(\mathbf{e}_j), \mathbf{f}_i \rangle_W &= \left\langle \sum_{k=1}^{m} a_{kj} \mathbf{f}_k, \mathbf{f}_i \right\rangle_W \\ &= \sum_{k=1}^{m} a_{kj} \langle \mathbf{f}_k, \mathbf{f}_i \rangle_W \\ &= \sum_{k=1}^{m} a_{kj} \delta_{ki} \\ &= a_{ij}. \end{aligned}$$

Example 2.4.10 says that the choice of an orthonormal basis E for an inner product space V relates the inner product of any two vectors $\mathbf{x}, \mathbf{y} \in V$ with the standard inner product on $\mathbb{K}^{n,1}$ of their coordinate vectors

$$\langle \mathbf{x}, \mathbf{y} \rangle_V = \left\langle \Phi_E^{-1}(\mathbf{x}), \Phi_E^{-1}(\mathbf{y}) \right\rangle_{\mathbb{K}^{n,1}} = \overline{[E, \mathbf{y}]}^\top [E, \mathbf{x}].$$

[6]'Operator' because this is the only setting in which it's meaningful to discuss eigenvectors.

Now observe that the following expressions hold for all $\mathbf{x}, \mathbf{y} \in V$:

$$\begin{aligned} T \text{ is orthogonal } &\iff \langle T(\mathbf{x}), T(\mathbf{y})\rangle_W = \langle \mathbf{x}, \mathbf{y}\rangle_V \\ &\iff \overline{[F, T(\mathbf{y})]}^\top [F, T(\mathbf{x})] = \overline{[E, \mathbf{y}]}^\top [E, \mathbf{x}] \\ &\iff \overline{(A\,[E, \mathbf{y}])}^\top (A\,[E, \mathbf{x}]) = \overline{[E, \mathbf{y}]}^\top [E, \mathbf{x}] \\ &\iff \overline{[E, \mathbf{y}]}^\top \overline{A}^\top A\,[E, \mathbf{x}]) = \overline{[E, \mathbf{y}]}^\top [E, \mathbf{x}] \\ &\iff \overline{[E, \mathbf{y}]}^\top (\overline{A}^\top A - I_n)[E, \mathbf{x}] = 0. \end{aligned}$$

In particular, if we choose $\mathbf{x}, \mathbf{y} \in V$ such[7] that $[E, \mathbf{x}] = \mathfrak{e}_i$ and $[E, \mathbf{y}] = \mathfrak{e}_j$, then the final equality reads that the $(i, j)^{\text{th}}$ entry of $\overline{A}^\top A - I_n$ is zero i.e. $\overline{A}^\top A = I_n$.

Definition 3.3.9 Let $A \in \mathbb{K}^{n,n}$.

- A is called **unitary** if $\overline{A}^\top A = I_n$.
- A is called **orthogonal** if $A^\top A = I_n$.

Remarks 3.3.10

- For a real matrix A, $\overline{A}^\top = A^\top$ so every real unitary matrix is orthogonal.
- Every unitary (or orthogonal) matrix is invertible.

We summarise the above exposition into a theorem.

Theorem 3.3.11 Let $E = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ and $F = (\mathbf{f}_1, \dots, \mathbf{f}_m)$ be orthonormal bases for complex inner product spaces V and W respectively. A linear map $T\colon V \to W$ is unitary $\iff$ $[F, T, E]$ is a unitary matrix.

If instead V and W are Euclidean spaces, and we replace 'unitary' with 'orthogonal', then $T\colon V \to W$ is orthogonal iff $[F, T, E]$ is an orthogonal matrix.

Remarks 3.3.12

- **Proposition 3.3.13** A complex matrix $U \in \mathbb{C}^{n,n}$ is unitary iff its columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ form an orthonormal set of vectors in $\mathbb{C}^{n,1}$ i.e. $\overline{\mathbf{a}_i}^\top \mathbf{a}_j = \delta_{ij}$ for all $i, j = 1, \dots, n$.

  Proof.

  $$\overline{A}^\top A = \begin{bmatrix} — & \overline{\mathbf{a}_1}^\top & — \\ & \vdots & \\ — & \overline{\mathbf{a}_n}^\top & — \end{bmatrix} \begin{bmatrix} | & & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{a}_1}^\top \mathbf{a}_1 & \overline{\mathbf{a}_1}^\top \mathbf{a}_2 & \cdots & \overline{\mathbf{a}_1}^\top \mathbf{a}_n \\ \overline{\mathbf{a}_2}^\top \mathbf{a}_1 & \overline{\mathbf{a}_2}^\top \mathbf{a}_2 & \cdots & \overline{\mathbf{a}_2}^\top \mathbf{a}_n \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\mathbf{a}_n}^\top \mathbf{a}_1 & \overline{\mathbf{a}_n}^\top \mathbf{a}_2 & \cdots & \overline{\mathbf{a}_n}^\top \mathbf{a}_n \end{bmatrix}$$

  and the $(i, j)^{\text{th}}$ entry of $\overline{A}^\top A$ is $\overline{\mathbf{a}_i}^\top \mathbf{a}_j$. The claim follows. ■

  An analogous statement holds for an orthogonal matrix.

- Since we define the angle θ between $\mathbf{x}, \mathbf{y} \in V$ in an inner product space by

  $$\cos(\theta) = \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|\mathbf{x}\|\|\mathbf{y}\|},$$

  we note that an orthogonal operator $T\colon V \to V$ also preserves angles because the equality defining the angle θ between $\mathbf{x}$ and $\mathbf{y}$ can equivalently be written as

  $$\cos(\theta) = \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{\langle T(\mathbf{x}), T(\mathbf{y})\rangle}{\|T(\mathbf{x})\|\|T(\mathbf{y})\|} \quad \text{since } T \text{ is orthogonal.}$$

[7] Recall that $\mathfrak{e}_i$ is the i^{th} standard basis vector of $\mathbb{K}^{n,1}$.

3.3.3 Unitary (Orthogonal) Diagonalisability

Let A be a complex normal matrix. We can think of it as the matrix representation of the linear operator $L_A\colon \mathbb{C}^{1,n} \to \mathbb{C}^{1,n}$ (defined by $\mathbf{x} \longmapsto A\mathbf{x}$) with respect to the standard orthonormal basis γ of $\mathbb{C}^{n,1}$. Since A is normal, so too is L_A, and by the complex spectral theorem, there exists an orthonormal basis E of $\mathbb{C}^{n,1}$ consisting of eigenvectors of A s.t. $D = [E, L_A, E]$ is a diagonal matrix. Let Q be the change-of-basis matrix from E to γ. Its columns are simply the eigenvectors in E. Thus, Q is a unitary matrix by **Proposition 3.3.13**. It follows that

$$\underbrace{[\gamma, L_A, \gamma]}_{=A} = Q\underbrace{[E, L_A, E]}_{=D}Q^{-1}$$

i.e. $A = QDQ^{-1}$ which means A is similar to a diagonal matrix. Since Q is unitary, $\overline{Q}^\top Q = I_n$ so we may replace Q^{-1} with $\overline{Q}^\top$ in the expression. We summarise the above into the following definition:

Definition 3.3.14 We say that a complex matrix A is **unitarily diagonalisable** if there exists a unitary matrix Q whose columns are an orthonormal basis of eigenvectors of A, and a diagonal matrix D of eigenvalues (corresponding to the basis of eigenvectors) of A s.t.

$$A = QDQ^{-1} = QD\overline{Q}^\top.$$

More generally, we have the following definition:

Definition 3.3.15 We say that a matrix A is **unitarily equivalent** to B if there exists a unitary matrix Q s.t. $A = QB\overline{Q}^\top$.

We can run a similar argument for a real symmetric matrix A. We can think of it as the matrix representation of linear operator $L_A\colon \mathbb{R}^{n,1} \to \mathbb{R}^{n,1}$ with respect to the standard orthonormal basis γ of $\mathbb{R}^{n,1}$. Since A is real symmetric (i.e. self-adjoint), L_A is self-adjoint. It follows from the real spectral theorem that $A = QDQ^{-1}$ for some orthogonal Q. We remark that Q is orthogonal so $Q^{-1} = Q^\top$.

Definition 3.3.16 We say that a real symmetric matrix A is **orthogonally diagonalisable** if there exists an orthogonal matrix Q whose columns are an orthonormal basis of eigenvectors of A, and a diagonal matrix D of eigenvalues (corresponding to the basis of eigenvectors) of A s.t.

$$A = QDQ^{-1} = QDQ^\top.$$

More generally:

Definition 3.3.17 A matrix A is **orthogonally equivalent** to B if there exists an orthogonal matrix Q s.t. $A = QBQ^\top$.

We can follow the reverse direction, and surmise that if a complex matrix $A \in \mathbb{C}^{n,n}$ is unitarily equivalent to a diagonal matrix D, then A is normal.

Proof. Let $A = QDQ^{-1}$ where Q is unitary. Then,

$$\begin{aligned} \overline{A}^\top A &= \overline{(QDQ^{-1})}^\top (QDQ^{-1}) & \qquad A\overline{A}^\top &= (QDQ^{-1})\overline{(QDQ^{-1})}^\top \\ &= \overline{Q^{-1}}^\top \overline{D}^\top \overline{Q}^\top QDQ^{-1} & &= QDQ^{-1}\overline{Q^{-1}}^\top \overline{D}^\top \overline{Q}^\top \\ &= Q\overline{D}^\top Q^{-1}QDQ^{-1} & &= QDQ^{-1}Q\overline{D}^\top Q^{-1} \\ &= Q\overline{D}^\top DQ^{-1} & &= QD\overline{D}^\top Q^{-1}. \end{aligned}$$

The diagonal matrices D and $\overline{D}^\top$ commute, and so A is normal. ■

Theorem 3.3.18 Let $A \in \mathbb{C}^{n,n}$. Then

$$A \text{ is normal} \iff A \text{ is unitarily equivalent to a diagonal matrix.}$$

An analogous characterisation holds for real symmetric matrices!

Theorem 3.3.19 Let $A \in \mathbb{R}^{n,n}$. Then A is symmetric iff A is orthogonally equivalent to a diagonal matrix.

I guess the key takeaway from the above developments is that, in the case of a matrix that is unitarily diagonalisable:

1. The unitary matrix $\overline{Q}^T = Q^{-1}$ rotates/reflects[8] the standard basis into the eigenbasis,
2. the diagonal matrix scales each coordinate independently by scale factor λ_i along each eigenvector $\mathbf{v}_i$,
3. and then the unitary matrix Q undoes the original transformation $Q^\top$ back to the original standard coordinate system we started off in.

This intuition came from a few worked examples.

Type these up.

[8]Or some combination of the two i.e. a rotation about an axis passing through (by virtue of linearity) the origin, and then a reflection by a plane for which the axis of rotation is normal to said plane. Such compositions are called 'improper reflections' or 'rotoreflections'.

Chapter 4

Orthogonal Projection, Spectral Theorem

4.1 Projections

Let V be a finite-dimensional inner product space, and W_1, W_2 be subspaces of V. Suppose that V admits the direct sum decomposition $V = W_1 \oplus W_2$, so any $\mathbf{x} \in V$ has the unique representation

$$\mathbf{x} = \underset{W_1}{\underset{\Cap}{\mathbf{x}_1}} + \underset{W_2}{\underset{\Cap}{\mathbf{x}_2}}.$$

This means that the assignment

$$\begin{aligned} T &: V \to V \\ &: \mathbf{x} \mapsto \mathbf{x}_1 \end{aligned}$$

is well-defined. Suppose that $\mathbf{y} \in V$ i.e. $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$ uniquely. Then for any $\alpha, \beta \in \mathbb{K}$

$$\begin{aligned} T(\alpha\mathbf{x} + \beta\mathbf{y}) &= T((\alpha\mathbf{x}_1 + \beta\mathbf{y}_1) + (\alpha\mathbf{x}_2 + \beta\mathbf{y}_2)) \\ &:= \alpha\mathbf{x}_1 + \beta\mathbf{y}_1) \\ &= \alpha\mathbf{x}_1 + \beta\mathbf{y}_1 \\ &=: \alpha T(\mathbf{x}) + \beta T(\mathbf{y}) \end{aligned}$$

so T is also linear. Any projection $S : V \to V$ from V on W_1 along W_2 must map $\mathbf{x} \longmapsto \mathbf{x}_1$. Since the decomposition of $\mathbf{x}$ is unique, there's only one value any projection can assign to each $\mathbf{x}$. Therefore, $S = T$. Thus, we may speak of **the** projection from V on W_1 along W_2.

Definition 4.1.1 In the above setting, linear operator $T : V \to V$ is called **the projection from V on W_1 along W_2** if $T(\mathbf{x}) = \mathbf{x}_1$.

Now we calculate the image and kernel of T. Note that if $\mathbf{x} \in V$ is in the image of T, then there exists some $\mathbf{u} \in V$ s.t. $T(\mathbf{u}) = \mathbf{x}$. Since $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ uniquely, $\mathbf{u}_1 = T(\mathbf{u}) = \mathbf{x}$ so $\mathbf{x} \in W_1$. Therefore, $\operatorname{image}(T) \subseteq W_1$. Conversely, if $\mathbf{x} \in W_1$ then $\mathbf{x} = \mathbf{x} + \mathbf{0}_V$ uniquely and so $T(\mathbf{x}) = T(\mathbf{x} + \mathbf{0}_V) = \mathbf{x}$. Thus, $\mathbf{x} \in \operatorname{image}(T)$ and so $W_1 \subseteq \operatorname{image}(T)$.

$$\therefore\ \operatorname{image}(T) = W_1.$$

Let $\mathbf{x} \in V$. Then $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ uniquely. Suppose that $\mathbf{x}$ is in the kernel of T. Then $\mathbf{0}_V = T(\mathbf{x}) = \mathbf{x}_1$ so $\mathbf{x} = \mathbf{0}_V + \mathbf{x}_1 \in W_2$. Conversely, suppose that $\mathbf{x} \in W_2$. Then $\mathbf{x} = 0_V + \mathbf{x}$ uniquely, and so $T(\mathbf{x}) = T(\mathbf{0}_V + \mathbf{x}) = \mathbf{0}_V$ i.e. $\mathbf{x} \in \ker(T)$.

$$\therefore\ \ker(T) = W_2.$$

4.1.1 Algebraic Characterisation of Projection

Lemma 4.1.2 Let $T : V \to V$ be a linear operator on a vector space V. Then $T^2 = T$ iff $V = \operatorname{image}(T) \oplus \ker(T)$, in which case T is the projection of V on $\operatorname{image}(T)$ along $\ker(T)$.

Proof.

$\Longleftarrow$ If $V = \operatorname{image}(T) \oplus \ker(T)$, then $T : V \to V$ is the projection of V on $W_1 = \operatorname{image}(T)$ along $W_2 = \ker(T)$. Any $\mathbf{x} \in V$ may be uniquely written as $\mathbf{x}_1 + \mathbf{x}_2$ where $\mathbf{x}_i \in W_i$. Then

$$T^2(\mathbf{x}) = T(T(\mathbf{x})) = T(\mathbf{x}_1) = \mathbf{x}_1 =: T(\mathbf{x}).$$

$\Longrightarrow$ Suppose that $T^2 = T$. We may write any $\mathbf{x} \in V$ as

$$\begin{aligned} \mathbf{x} &= \mathbf{x} + (T(\mathbf{x}) - T(\mathbf{x})) \\ &= T(\mathbf{x}) + (\mathbf{x} - T(\mathbf{x})). \end{aligned}$$

Observe that

$$\begin{aligned} T(\mathbf{x} - T(\mathbf{x})) &= T(\mathbf{x}) - T^2(\mathbf{x}) \\ &= T(\mathbf{x}) - T(\mathbf{x}) \quad \text{since } T^2 = T \\ &= \mathbf{0}_V \end{aligned}$$

and so $\mathbf{x} - T(\mathbf{x})$ is in the kernel of T. Also, $T(\mathbf{x})$ is clearly in the image of T, so

$$\mathbf{x} = \underset{\substack{\Cap \\ \operatorname{im}(T)}}{T(\mathbf{x})} + \underset{\substack{\Cap \\ \ker(T)}}{(\mathbf{x} - T(\mathbf{x}))}.$$

Thus, $V = \operatorname{image}(T) + \ker(T)$.

For uniqueness, suppose that $\mathbf{x} \in \operatorname{image}(T) \cap \ker(T)$. Since $\mathbf{x} \in \operatorname{image}(T)$, there exists some $\mathbf{u} \in V$ s.t. $T(\mathbf{u}) = \mathbf{x}$. Applying T once more gives

$$T(\mathbf{x}) = T(T(\mathbf{u})) = T^2(\mathbf{u}) = T(\mathbf{u}) = \mathbf{x}. \tag{$\mathrm{id}_{\mathrm{Proj.}}$}$$

Since $\mathbf{x} \in \ker(T)$, $T(\mathbf{x}) = \mathbf{0}_V$, and we combine this with the former equality to obtain

$$\mathbf{0}_V = T(\mathbf{x}) = T^2(\mathbf{u}) = T(\mathbf{u}) = \mathbf{x}$$

i.e. $\operatorname{image}(T) \cap \ker(T) \subseteq \{\mathbf{0}_V\}$. Thus, the Minkowski sum is direct.

■

Definition 4.1.3 Therefore, a linear operator $T \colon V \to V$ is **the projection of V onto its image** iff $T^2 = T$.

Remarks 4.1.4 **Equation** ($\mathrm{id}_{\mathrm{Proj.}}$) demonstrates a fact that is very useful in computation — a projection acts as the identity on its image.

4.1.2 When is a projection uniquely determined by its range?

In general, the image W_1 of a projection does not uniquely characterise a projection. We can see how this works for a simple example.

Example 4.1.5 Let

$$W_1 = \operatorname{span}\left(\begin{bmatrix}1\\0\end{bmatrix}\right), \ W_2 = \operatorname{span}\left(\begin{bmatrix}0\\1\end{bmatrix}\right), \text{ and } W_3 = \operatorname{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right).$$

Then it's easy to demonstrate that $V = W_1 \oplus W_2 = W_1 \oplus W_3$.

We can write any

$$\mathbb{R}^2 \ni \mathbf{v} = \begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix} = (x-y)\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}1\\1\end{bmatrix}$$

so certainly $V = W_1 + W_2 = W_1 + W_3$. What remains to show is the trivial intersection condition for uniqueness:

- $\begin{bmatrix}x\\y\end{bmatrix} \in W_1 \cap W_3 \iff \exists \alpha, \beta \in \mathbb{R}$ s.t. $\mathbf{v} = \begin{bmatrix}x\\y\end{bmatrix} = \alpha\begin{bmatrix}1\\0\end{bmatrix} = \beta\begin{bmatrix}1\\1\end{bmatrix}$

  i.e. $\alpha = \beta$ and $0 = \beta$. Therefore, $\mathbf{v} = \mathbf{0}_V$.

- $\begin{bmatrix} x \\ y \end{bmatrix} \in W_1 \cap W_2 \iff \exists \alpha, \beta \in \mathbb{R} \text{ s.t. } \mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

  i.e. $\alpha = 0$ and $0 = \beta$. Therefore, $\mathbf{v} = \mathbf{0}_V$.

The projection of V on W_1 along W_2 (the proverbial "x-axis") is the linear operator $T_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

with matrix representation (w.r.t. the standard ordered basis of $\mathbb{R}^2$)

$$[E_{\text{std}}, T_2, E_{\text{std}}] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The projection of V on W_1 along W_3 (the proverbial line "$y = x$") is the linear operator $T_3 \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T_3\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ 0 \end{bmatrix}$$

with matrix representation

$$[E_{\text{std}}, T_3, E_{\text{std}}] = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

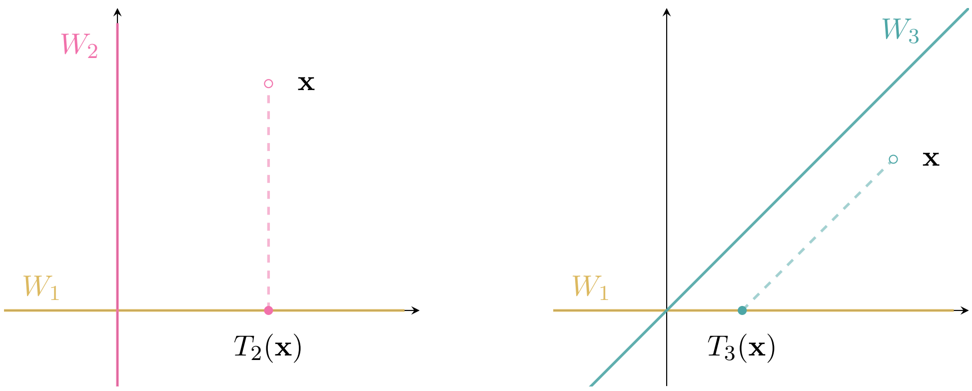


Figure 4.1: Visualisations of the projections T_2 and T_3 along the subspaces W_2 and W_3, respectively.

Clearly these two projection maps are different, so specifying a different complementary subspace W_2 to W_1 determines a different projection map on W_1. Thus, the pair (W_1, W_2) of complementary subspaces of V uniquely determines a projection map from $V = W_1 \oplus W_2$ on W_1 along W_2. ▶

4.2 Orthogonal Projection

At this point, a sensible question to ask is when the image uniquely defines a projection i.e. when does $W_1 = \text{image}(T)$ completely determine $W_2 = \ker(T)$.

The answer to this is precisely when V has an inner product structure. By **Theorem 2.4.19**, if W_1 is a finite-dimensional subspace of an inner product space V, then there is a canonical choice of complement W_2 in V for W_1 — the orthogonal complement $W_2 = W_1^\perp$ which satisfies

$$V = W_1 \oplus W_1^\perp.$$

Now W_1 alone uniquely defines the projection P of V on W_1 along $W_1^\perp$ defined (in the notation of **Section 2.4.3** i.e. $\mathbf{y} = \mathbf{u} + \mathbf{z}$ uniquely) by $P(\mathbf{y}) = \mathbf{u}$.

The conditions $W_1 = \text{image}(T)$ and $\ker(T) = W_2 = W_1^\perp$ imply, by **Exercise 3 (c)**, that $(\ker(T))^\perp = (W_1^\perp)^\perp = W_1 = \text{image}(T)$. Thus, the image and kernel W_1 and W_2 of T are orthogonally complementary. The projection map from V on a finite-dimensional W_1 along $W_1^\perp$ is called **the orthogonal projection from V on W_1**.

In general, for any subspace W of an inner product space V, we can only claim by **Exercise 3 (b)** that $(W^\perp)^\perp \subseteq W$. To guarantee that the image and kernel of a projection are orthogonally complementary, we adopt the following definition.

Definition 4.2.1 A projection[1] $P\colon V \to V$ onto its image is called an **orthogonal projection** if both $(\ker(P))^\perp = \text{image}(P)$ and $(\text{image}(P))^\perp = \ker(P)$.

4.2.1 Algebraic characterisation of orthogonal projections

Lemma 4.2.2 Let V be an inner product space, and T be a linear operator on V. Then T is an orthogonal projection iff T has an adjoint $T^\dagger$ and $T^2 = T = T^\dagger$.

Proof.

$\Longrightarrow$ Suppose that T is an orthogonal projection. Since T is a projection, we know that

- $T^2 = T$
- $V = \text{image}(T) \oplus \ker(T)$
- $\ker(T) = (\text{image}(T))^\perp$ and $\text{image}(T) = (\ker(T))^\perp$

so we only need to show that $T^\dagger$ exists, and that T is self-adjoint.

Let $\mathbf{v}, \mathbf{w} \in V$ which may be uniquely written as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ respectively. Therefore,

$$\begin{aligned} \langle \mathbf{v}, T(\mathbf{w}) \rangle &= \langle \mathbf{v}_1 + \mathbf{v}_2, T(\mathbf{w}_1 + \mathbf{w}_2) \rangle \\ &= \langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 \rangle \\ &= \langle \mathbf{v}_1, \mathbf{w}_1 \rangle + \langle \mathbf{v}_2, \mathbf{w}_1 \rangle \\ &= \langle \mathbf{v}_1, \mathbf{w}_1 \rangle + 0 \end{aligned}$$

since $\mathbf{v}_2 \in \ker(T)$ and $\mathbf{w}_1 \in \text{image}(T) = (\ker(T))^\top$, and

$$\begin{aligned} \langle T(\mathbf{v}), \mathbf{w} \rangle &= \langle T(\mathbf{v}_1 + \mathbf{v}_2), \mathbf{w}_1 + \mathbf{w}_2 \rangle \\ &= \langle \mathbf{v}_1, \mathbf{w}_1 + \mathbf{w}_2 \rangle \\ &= \langle \mathbf{v}_1, \mathbf{w}_1 \rangle + \langle \mathbf{v}_1, \mathbf{w}_2 \rangle \\ &= \langle \mathbf{v}_1, \mathbf{w}_1 \rangle + 0 \end{aligned}$$

since $\mathbf{v}_1 \in \text{image}(T)$ and $\mathbf{w}_2 \in \ker(T) = (\text{image}(T))^\top$, from which it follows that

$$\langle T(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}_1, \mathbf{w}_1 \rangle = \langle \mathbf{v}, T(\mathbf{w}) \rangle \quad \text{for all } \mathbf{v}, \mathbf{w} \in V$$

i.e. $T^\dagger$ exists and $T = T^\dagger$.

$\Longleftarrow$ Conversely, suppose that T has an adjoint, and satisfies $T^2 = T = T^\dagger$. We wish to show that T is an orthogonal projection.

Since $T^2 = T$, it's a projection from V on its image along its kernel. What remains to show is mutual orthogonality between the image and kernel of T. Let $\mathbf{x} \in \text{image}(T)$ and $\mathbf{y} \in \ker(T)$. Since $\mathbf{x} \in \text{image}(T)$ and T acts like the identity on its image, we observe that $\mathbf{x} = T(\mathbf{x}) = T^\dagger(\mathbf{x})$ and so

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= \langle T^\dagger(\mathbf{x}), \mathbf{y} \rangle \\ &= \langle \mathbf{x}, T(\mathbf{y}) \rangle \\ &= \langle \mathbf{x}, \mathbf{0}_V \rangle \\ &= 0 \end{aligned}$$

[1] Recall that $P^2 = P$ iff the direct sum decomposition $V = \text{image}(T) \oplus \ker(T)$ holds. Furthermore, note that this definition does not require that $W_1 = \text{image}(T)$ is finite-dimensional.

Therefore, $\mathbf{x} \in (\ker(T))^\perp$ and so $\text{image}(T) \subseteq (\ker(T))^\perp$.

Now let $\mathbf{y} \in (\ker(T))^\perp$. We must show that $\mathbf{y} \in \text{image}(T)$ i.e. that $T(\mathbf{y}) = \mathbf{y}$.

$$\begin{aligned}
\|\mathbf{y} - T(\mathbf{y})\|^2 &:= \langle \mathbf{y} - T(\mathbf{y}), \mathbf{y} - T(\mathbf{y}) \rangle \\
&= \langle \mathbf{y}, \underbrace{\mathbf{y} - T(\mathbf{y})}_{\in \ker(T)} \rangle - \langle T(\mathbf{y}), \mathbf{y} - T(\mathbf{y}) \rangle \\
&= 0 - \langle T(\mathbf{y}), \mathbf{y} - T(\mathbf{y}) \rangle \\
&= -\Big\langle \mathbf{y}, T^\dagger(\mathbf{y} - T(\mathbf{y})) \Big\rangle \\
&= -\langle \mathbf{y}, T(\mathbf{y} - T(\mathbf{y})) \rangle \\
&= -\langle \mathbf{y}, \mathbf{0}_V \rangle \\
&= 0
\end{aligned}$$

Therefore, $\mathbf{y} = T(\mathbf{y})$ i.e. $\mathbf{y} \in \text{image}(T)$, and so $(\ker(T))^\perp \subseteq \text{image}(T)$.

We conclude that $(\ker(T))^\perp = \text{image}(T)$. Now take the orthogonal complement of this equality of sets to obtain

$$(\text{image}(T))^\perp = ((\ker(T))^\perp)^\perp \supseteq \ker(T) \quad \text{by } \textbf{Exercise 3 (b)}.$$

All that remains is to demonstrate the reverse inclusion: Suppose that $\mathbf{x} \in (\text{image}(T))^\perp$. Let $\mathbf{y} \in V$. Then

$$\begin{aligned}
\langle T(\mathbf{x}), \mathbf{y} \rangle &= \langle \mathbf{x}, T^\dagger(\mathbf{y}) \rangle \\
&= \langle \mathbf{x}, T(\mathbf{y}) \rangle \\
&= 0
\end{aligned}$$

since $\mathbf{x} \in (\text{image}(T))^\perp$. Therefore $T(\mathbf{x}) = \mathbf{0}_V$, and so $\mathbf{x} \in \ker(T)$. Thus, $\ker(T) = (\text{image}(T))^\perp$ which concludes the proof.

■

4.2.2 Matrix representation

Let V be a finite-dimensional inner product space, and T be the projection from V on its image $W_1 = \text{image}(T)$. Thus, $V = \text{image}(T) \oplus \ker(T)$. Denote $\ker(T)$ by W_2. Suppose that $\dim(W) = k$ and $\dim(V) = n$. We may choose bases $E_1 = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ for $\text{image}(T)$, and $E_2 = (\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n)$ for W_2 s.t. their union E is a basis for V. Since $\mathbf{v}_i \in W = \text{image}(T)$ for $1 \leqslant i \leqslant k$, and T acts on its image as the identity, we have that $T(\mathbf{v}_i) = \mathbf{v}_i$ for $1 \leqslant i \leqslant k$. For $i > k$, $\mathbf{v}_i \in W^\perp$ and so $T(\mathbf{v}_i) = \mathbf{0}_V$. This means that the matrix representation of the projection from V on its image with respect to the basis E is

$$[E, T, E] = \begin{bmatrix} I_k & 0 \\ 0 & 0_{n-k} \end{bmatrix}.$$

In the case of an orthogonal projection P from V on its image, the basis may be chosen so that it's orthonormal, giving the same matrix representation.

4.3 Spectral Decomposition

We've already proven the complex (resp. real) spectral theorem which characterises a normal (resp. self-adjoint) linear operator $T\colon V \to V$ by the admission of an orthonormal basis for V consisting of eigenvectors of T. Now we may use the theory of projections in order to present a basis-free canonical decomposition of such a T into projections onto its eigenspaces.

Theorem 4.3.1 (Spectral Decomposition Theorem Part 1) Let T be a linear operator on a finite-dimensional inner product space V over $\mathbb{K}$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Assume that

- T is normal if $\mathbb{K} = \mathbb{C}$, or
- T is self-adjoint if $\mathbb{K} = \mathbb{R}$.

For each λ_i, let $W_i = \ker(T - \lambda_i \operatorname{id}_V)$ be the eigenspace of T corresponding to λ_i. Then:

(a) $V = \bigoplus_{i=1}^{k} W_i$

(b) Denote by $W_{[i]} := \bigoplus_{\substack{1 \leqslant j \leqslant k \\ j \neq i}} W_j$. Then $W_i^\perp = W_{[i]}$.

Proof.

(a) This follows directly from the complex (if $\mathbb{K} = \mathbb{C}$) or real (if $\mathbb{K} = \mathbb{R}$) spectral theorems because T-diagonalisable is equivalent to $V = W_1 \oplus \ldots \oplus W_k$.

(b) Let $\mathbf{x} = W_i$ and $\mathbf{y}_j \in W_j$ for some $j \neq i$. Then $\langle \mathbf{x}, \mathbf{y}_j \rangle = 0$ since eigenvectors corresponding to distinct eigenvalues are orthogonal. Any $\mathbf{y} \in W_{[i]}$ can be written uniquely as $\mathbf{y} = \mathbf{y}_1 + \ldots + \mathbf{y}_{i-1} + \mathbf{y}_{i+1} + \mathbf{y}_k$ where $\mathbf{y}_j \in W_j$. It follows by linearity that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{\substack{1 \leqslant j \leqslant k \\ j \neq i}} \langle \mathbf{x}, \mathbf{y}_i \rangle = 0$$

and $\mathbf{x}$ was arbitrary in W_i so $\mathbf{y} \in W_i^\perp$. Thus, $W_{[i]} \subseteq W_i^\perp$. By the dimension formula for direct sums, we have that:

- $\dim(W_{[i]}) = \sum_{\substack{1 \leqslant j \leqslant k \\ j \neq i}} \dim(W_j)$
- $\dim(V) = \sum_{1 \leqslant j \leqslant k} \dim(W_j)$.

Using these two equations, we conclude that

$$\dim(W_{[i]}) = \dim(V) - \dim(W_i).$$

Furthermore, since W_i is a subspace of V, it follows from **Theorem 2.4.23 (c)** that $\dim(V) = \dim(W_i) + \dim(W_i^\perp)$ i.e.

$$\dim(W_i^\perp) = \dim(V) - \dim(W_i).$$

Since $W_{[i]} \subseteq W_i^\perp$ are subspaces of V with the same dimension, they are equal.

■

Parts (a) and (b) tell us that $W_{[i]} = W_i^\perp$. Since V is finite-dimensional, it holds that

$$(W_{[i]})^\perp = (W_i^\perp)^\perp = W_i$$

so each eigenspace W_i and its complement in V are orthogonal to each other. Thus, the decomposition $V = W_1 \oplus \ldots \oplus W_k$ is orthogonal in that sense.

The first part of my spectral decomposition theorem **Theorem 4.3.1** establishes the decomposition of V into "orthogonal" eigensubspaces W_i by leveraging the real/complex spectral theorem and the underlying inner product space's properties:

> Suppose there are k distinct eigenvalues, this gives k eigenspaces $W_1, \ldots, W_k$, the real/complex spectral theorems give the decomposition $V = W_1 \oplus \cdots \oplus W_k$, and then we observe (in either case; real or complex) that distinct eigenvalues of a normal operator (self-adjoint means trivially normal) correspond to orthogonal eigenvectors. Then we argue by dimension arguments and orthogonal complements.

However, there is a method to establish this with a linear operator acting on just a vector space. Indeed, Part 2 of the spectral decomposition theorem (**Theorem 4.3.2**) makes no direct reference to inner products.

Thank you to Eric Wayman for pointing this out to me. I cite a few of the key points verbatim (modulo using $\mathbb{K}$ instead of $\mathcal{F}$ for a field etc.) that Professor David Bruce Surowski uses in his notes *Advanced Linear Algebra* [3] to develop this alternative pathway:

> Let V be a vector space over $\mathbb{K}$.
>
> **Theorem** (**2.2.10** [3, p. 63] Primary Decomposition Theorem, Part I) Let $T\colon V \to V$, $\dim(V) < \infty$, $m_T(x) = p_1(x)^{e_1} \cdots p_k(x)^{e_k}$, where $p_1(x)$, $\ldots$, $p_k(x)$ are distinct monic irreducible polynomials in $\mathbb{K}[x]$. For each $i = 1, \ldots, k$, set
> $$V_i = \ker(p_i(T)^{e_i}).$$
> Then each V_i is T-invariant and
> $$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k.$$
> Furthermore, if
> $$T|_{V_i} \colon V_i \to V_i$$
> is the restriction of T to V_i, then
> $$m_{T|_{V_i}}(x) = p_i(x)^{e_i}, \quad i = 1, \ldots, k.$$
>
> **Definition** ([3, p. 65]) A linear operator $P\colon V \to V$ is called an idempotent if and only if $P^2 = P$. Note that any idempotent must be a root of the polynomial $x(x-1)$ and hence must be diagonalisable (with all eigenvalues being 0 or 1). A family $P_1, P_2, \ldots, P_k$ of idempotents is called orthogonal if $P_i \circ P_j = 0$ whenever $i \neq j$.
>
> **Theorem** (**2.2.11** [3, p. 65] Primary Decomposition Theorem, Part II) Let $T\colon V \to V$, $\dim(V) < \infty$, $m_T(x) = p_1(x)^{e_1} \cdots p_k(x)^{e_k}$, where $p_1(x)$, $\ldots$, $p_k(x)$ are distinct monic irreducible polynomials in $\mathbb{K}[x]$. Then there exist orthogonal idempotents $P_1, P_2, \ldots, P_k$ commuting with T such that
> $$V = P_1(V) \oplus P_2(V) \oplus \cdots \oplus P_k(V);$$
> furthermore, we have
> $$P_i(V) = \ker(p_i(T)^{e_i}), \quad i = 1, 2, \ldots, k.$$
>
> **Corollary** (**2.2.11.1** [3, p. 66] Spectral Decomposition) Let $T\colon V \to V$ **be a diagonalisable linear operator**. Then there exist orthogonal idempotents P_1, $\ldots$, P_k, each commuting with T, and scalars λ_1, $\ldots$, λ_k (the eigenvalues of T) such that
> $$T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_k P_k.$$

I believe the condition that T **is diagonalisable** (which is slightly more general than the case of the orthogonal diagonalisability assumed in my Part 1 where we capitalise on the spectral theorems) reduces these $V_i = \ker(p_i(T)^{e_i})$ to my eigenspaces $W_i = \ker(T - \lambda_i \operatorname{id}_V)$. As for the more general V_i, I'm not entirely sure what's going on but it's probably something to do with minimal polynomials and Jordan-something-or-others.

Theorem 4.3.2 (**Spectral Decomposition Theorem Part 2**) Let P_i denote the orthogonal projection of V on W_i along $W_{[i]} = W_i^{\perp}$. Then

(c) $P_i \circ P_j = \delta_{ij} P_i$ for $i, j = 1, \ldots, k$.

(d) $\mathrm{id}_V = P_1 + \ldots + P_k$

(e) $T = \lambda_1 P_1 + \ldots + \lambda_k P_k$.

Proof.

(c) By (a), any $\mathbf{x} \in V$ may be uniquely written as $\mathbf{x} = \mathbf{x}_1 + \ldots + \mathbf{x}_k$ where $\mathbf{x}_i \in W_i$ for all i.

$$\begin{aligned} P_i(P_j(\mathbf{x})) = P_i(\mathbf{x}_j) &= \begin{cases} \mathbf{x}_i & \text{if } j = i \\ \mathbf{0}_V & \text{if } j \neq i \end{cases} \\ &= \begin{cases} P_i(\mathbf{x}) & \text{if } j = i \\ \mathbf{0}_V & \text{if } j \neq i \end{cases} \\ &= \delta_{ij} P_i(\mathbf{x}) \end{aligned}$$

(d) Since P_i is the orthogonal projection of V on W_i along $W_{[i]} = W_i^{\perp}$, we have by definition $P_i(\mathbf{x}) = \mathbf{x}_i$. Thus, it follows that for any $\mathbf{x} \in V$:

$$\begin{aligned} \mathrm{id}_V(\mathbf{x}) = \mathbf{x} &= \mathbf{x}_1 + \ldots + \mathbf{x}_k \\ &= P_1(\mathbf{x}) + \ldots + P_k(\mathbf{x}). \end{aligned}$$

(e) This also follows easily. Any $\mathbf{x} \in V$ has the unique representation $\mathbf{x} = \mathbf{x}_1 + \ldots + \mathbf{x}_k$ where $\mathbf{x}_i \in W_i = \ker(T - \lambda_i \,\mathrm{id}_V)$ i.e. $T(\mathbf{x}_i) = \lambda_i \mathbf{x}_i$. Thus,

$$\begin{aligned} T(\mathbf{x}) &= T(\mathbf{x}_1) + \ldots + T(\mathbf{x}_k) \\ &= \lambda_1 \mathbf{x}_1 + \ldots + \lambda_k \mathbf{x}_k \\ &= \lambda_1 P_1(\mathbf{x}) + \ldots + \lambda_k P_k(\mathbf{x}) \\ &= (\lambda_1 P_1 + \ldots + \lambda_k P_k)(\mathbf{x}). \end{aligned}$$

■

CHAPTER 5

Modulus & Singular Value Decomposition

Thus far, we've demonstrated the real and complex spectral theorems (for self-adjoint, and normal operators, respectively) that characterise a linear operator $T\colon V \to V$ in terms of an orthonormal basis for V consisting of eigenvectors of T and their corresponding eigenvalues. This is a very niche use-case for anything practical. If we pick a matrix from the collection $\mathbb{K}^{m,n}$, we'd need to strike oil to have a matrix that's square *and* Hermitian.

We now exit the world of linear operators to discuss linear maps $T\colon V \to W$ (but really we'll be using the theory of linear operators because that's all we have) between finite-dimensional inner product spaces $(V, \langle\,\cdot\,,\,\cdot\,\rangle_V)$ and $(W, \langle\,\cdot\,,\,\cdot\,\rangle_W)$ over $\mathbb{K}$. The goal of this chapter is to arrive at a theorem, comparable to the spectral theorem, that applies to any linear operator $T \in \mathrm{Hom}_{\mathbb{K}}(V; W)$.

We can't discuss the notion of eigenvalues and eigenvectors for a linear map that doesn't map V into itself. Instead, we can ask the more primitive question of how T scales any vector $\mathbf{v}$ in norm

$$\|T(\mathbf{v})\|_W^2 := \langle T(\mathbf{v}), T(\mathbf{v})\rangle_W = \langle \mathbf{v}, T^\dagger(T(\mathbf{v}))\rangle_V$$

and so the **Hermitian square** $T^\dagger \circ T\colon V \to V$ is a map that captures how T scales $\mathbf{v}$. Indeed, it's a linear operator that is self-adjoint because

$$(T^\dagger \circ T)^\dagger = T^\dagger \circ T^{\dagger\dagger} = T^\dagger \circ T.$$

Thus, its eigenvalues are real, and if $\mathbf{v} \neq \mathbf{0}_V$ is an eigenvector of $T^\dagger \circ T$ with eigenvalue λ, then

$$\begin{aligned}\|T(\mathbf{v})\|_W^2 &= \langle \mathbf{v}, T^\dagger(T(\mathbf{v}))\rangle_V \\ &= \overline{\lambda}\langle \mathbf{v}, \mathbf{v}\rangle_V \\ &= \lambda\|\mathbf{v}\|^2.\end{aligned}$$

It would be wonderful if λ were non-negative, for we could then take the square root of the above expression to obtain $\|T(\mathbf{v})\| = \sqrt{\lambda}\|\mathbf{v}\|$ i.e. $\sqrt{\lambda}$ would be the scale factor by which T stretches any eigenvector $\mathbf{v}$ of $T^\dagger \circ T$ corresponding to the eigenvalue λ. It turns out that $T^\dagger \circ T$ is positive semidefinite, and this can be equivalently stated in terms of the eigenvalues of $T^\dagger \circ T$ all being non-negative. We'll prove that intermediate fact (and a few others that will come in handy for manipulating $T^\dagger \circ T$) before continuing this line of thought in **Section 5.0.3**.

5.0.1 ADJOINT FACTS

Lemma 5.0.1 (Exercise 15, Sec. 6.3 [1])

(c) $\mathrm{rank}(T) = \mathrm{rank}(T^\dagger)$.

Proof. By the rank-nullity theorem on $T^\dagger\colon W \to V$,

$$\dim(W) = \mathrm{rank}(T^\dagger) + \mathrm{nullity}(T^\dagger).$$

We're looking for $\mathrm{rank}(T^\dagger)$, and $\dim(W)$ is something we can't change, so the only thing left to play with is $\mathrm{nullity}(T^\dagger) = \dim(\ker(T^\dagger))$. What does $\ker(T^\dagger)$ look like? Let $\mathbf{y} \in \ker(T^\dagger)$, then $T^\dagger(\mathbf{y}) = \mathbf{0}_V$. Now observe that

$$\begin{aligned}\mathbf{y} \in \ker(T^\dagger) &\iff T^\dagger(\mathbf{y}) = \mathbf{0}_V \\ &\iff \underbrace{\langle \mathbf{x}, T^\dagger(\mathbf{y})\rangle_W}_{=\langle T(\mathbf{x}), \mathbf{y}\rangle_V} = 0 \quad \text{for all } \mathbf{x} \in V \\ &\iff \langle T(\mathbf{x}), \mathbf{y}\rangle_V = 0 \quad \text{for all } \mathbf{x} \in V \\ &\iff \mathbf{y} \in (\mathrm{image}(T))^\perp\end{aligned}$$

Thus, $\ker(T^\dagger) = (\mathrm{image}(T))^\perp$. So $\mathrm{nullity}(T^\dagger) = \dim(\ker(T^\dagger)) = \dim((\mathrm{image}(T))^\perp)$.

Since $\mathrm{image}(T)$ is a subspace of W, the dimension formula gives

$$\begin{aligned}\dim(W) &= \dim(\mathrm{image}(T)) + \dim((\mathrm{image}(T))^\perp)\\ &= \dim(\mathrm{image}(T)) + \dim(\ker(T^\dagger))\\ &= \mathrm{rank}(T) + \mathrm{nullity}(T^\dagger).\end{aligned}$$

Now we combine the dimension formula with the rank-nullity formula at the start of this proof to conclude that

$$\mathrm{rank}(T) + \cancel{\mathrm{nullity}(T^\dagger)} = \dim(W) = \mathrm{rank}(T^\dagger) + \cancel{\mathrm{nullity}(T^\dagger)}$$

Therefore, $\mathrm{rank}(T) = \mathrm{rank}(T^\dagger)$. ■

5.0.2 Facts About the Hermitian Square

The property of positive (semi)definiteness allows us to characterise the eigenvalues of the self-adjoint operator $T^\dagger \circ T$. Note that if T is self-adjoint, then

$$\overline{\langle T(\mathbf{x}), \mathbf{x}\rangle} = \langle \mathbf{x}, T(\mathbf{x})\rangle = \langle T^\dagger(\mathbf{x}), \mathbf{x}\rangle = \langle T(\mathbf{x}), \mathbf{x}\rangle$$

where the final equality follows from self-adjointness. This means that the quantity $\langle T(\mathbf{x}), \mathbf{x}\rangle \in \mathbb{R}$.

Definition 5.0.2 A linear operator $T\colon V \to V$ on a finite-dimensional inner product space V is called:

- **positive semidefinite** if T is self-adjoint and

$$\langle T(\mathbf{x}), \mathbf{x}\rangle \geqslant 0 \text{ for all } \mathbf{x} \in V,$$

- **positive definite** if T is self-adjoint and

$$\langle T(\mathbf{x}), \mathbf{x}\rangle > 0 \text{ for all } \mathbf{x} \in V \setminus \{\mathbf{0}_V\}.$$

Remarks 5.0.3

- We denote T being positive semidefinite by $T \geqslant 0$, and T being positive definite by $T > 0$.
- We now derive the analogous statement of positive definiteness for the matrix of T. Let E be an orthonormal basis for V, and $A = [E, T, E]$. Then

$$\begin{aligned}T \geqslant 0 &\iff T = T^\dagger \quad \text{and} \quad \langle T(\mathbf{v}), \mathbf{v}\rangle \geqslant 0 \text{ for all } \mathbf{v} \in V\\ &\iff A = \overline{A}^\top \quad \text{and} \quad \langle T(\mathbf{v}), \mathbf{v}\rangle = \overline{[E, \mathbf{v}]}^\top A\,[E, \mathbf{v}] \geqslant 0 \quad \text{for all } \mathbf{v} \in V.\end{aligned}$$

  We give a name to this property of A.

Definition 5.0.4 An $n \times n$ complex matrix A is called:

- **positive semidefinite** if $A = \overline{A}^\top$ and $\overline{\mathbf{x}}^\top A\mathbf{x} \geqslant 0$ for all $\mathbf{x} \in \mathbb{C}^{n,1}$,
- **positive definite** if $A = \overline{A}^\top$ and $\overline{\mathbf{x}}^\top A\mathbf{x} > 0$ for all non-zero $\mathbf{x} \in \mathbb{C}^{n,1}$.

Remarks 5.0.5

- We denote A being positive semidefinite by $A \succcurlyeq 0$, and A being positive definite by $A \succ 0$.
- A more general definition of $A \in \mathbb{C}^{n,n}$ being positive semidefinite requires that $\Re(\overline{\mathbf{x}}^\top A\mathbf{x}) \geqslant 0$. This does not require self-adjointess of A.

Lemma 5.0.6 Let T be a self-adjoint operator on a finite-dimensional inner product space V. Then $T \geqslant 0$ (resp. $T > 0$) iff all eigenvalues of T are non-negative (resp. positive).

Proof. Since T is self-adjoint, the real spectral theorem tells us that there exists an orthonormal basis $E = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of V consisting of eigenvectors of T (i.e. $T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$), and $[E, T, E] = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. A diagonal matrix is positive semidefinite (resp. positive definite) iff all its diagonal entries are non-negative (resp. positive) which concludes the proof. ■

Lemma 5.0.7 (Exercise 18, Sec. 6.4 [1])

(a) The Hermitian square $T^\dagger \circ T$ and $T \circ T^\dagger$ are positive semidefinite.

(b) $\operatorname{rank}(T^\dagger \circ T) = \operatorname{rank}(T) = \operatorname{rank}(T \circ T^\dagger)$.

Proof.

(a) Note that $(T^\dagger \circ T)^\dagger = T^\dagger \circ (T^\dagger)^\dagger = T^\dagger \circ T$, and similarly, $(T \circ T^\dagger)^\dagger = T \circ T^\dagger$ so both are self-adjoint operators on V and W respectively. By the previous exercise, it suffices to show that every eigenvalue is non-negative to conclude that $T^\dagger \circ T$ is positive semidefinite. Let λ be an eigenvalue of $T^\dagger \circ T$ with corresponding eigenvector $\mathbf{v} \neq \mathbf{0}_V$ i.e. $T^\dagger(T(\mathbf{v})) = \lambda \mathbf{v}$. Hence,

$$\lambda \|\mathbf{v}\|^2 = \langle T^\dagger(T(\mathbf{v})), \mathbf{v} \rangle = \langle T(\mathbf{v}), T(\mathbf{v}) \rangle = \|T(\mathbf{v})\|^2 \quad \text{i.e. } \lambda = \frac{\|T(\mathbf{v})\|^2}{\|\mathbf{v}\|^2}.$$

Since $\mathbf{v}$ is an eigenvector, $\|\mathbf{v}\| > 0$. The numerator is always non-negative. Thus, $\lambda \geqslant 0$ i.e. $T^\dagger \circ T$ is positive semidefinite.

We can also figure out the conditions under which $T^\dagger \circ T$ is positive definite. Namely, we need that $\lambda = \|T(\mathbf{v})\|^2 / \|v\|^2 > 0$ which is equivalent to showing that $\|T(\mathbf{v})\| > 0$ i.e. $T(\mathbf{v}) \neq \mathbf{0}_V$. This holds iff T has trivial kernel i.e. T is injective. Thus, $T^\dagger \circ T$ is positive definite iff T is injective.

(b) Proof sketch:

- We've already proven that $\operatorname{rank}(T) = \operatorname{rank}(T^\dagger)$.
- By the rank-nullity theorem for $T^\dagger \circ T$

$$\dim(V) = \operatorname{rank}(T^\dagger \circ T) + \dim(\ker(T^\dagger \circ T)),$$

and by the rank-nullity theorem for T

$$\dim(V) = \operatorname{rank}(T) + \dim(\ker(T)).$$

  - If we can show that $\ker(T) = \ker(T^\dagger \circ T)$, then we can combine the above equations to prove that $\operatorname{rank}(T^\dagger \circ T) = \operatorname{rank}(T)$.

Lemma 5.0.8 $\ker(T) = \ker(T^\dagger \circ T)$.

Proof. Let $\mathbf{x} \in \ker(T)$ i.e. $T(\mathbf{x}) = \mathbf{0}_V$. Then $T^\dagger(T(\mathbf{x})) = T^\dagger(\mathbf{0}_W) = \mathbf{0}_V$ by the linearity of $T^\dagger$. Therefore, $\mathbf{x} \in \ker(T^\dagger \circ T)$.

For the reverse inclusion, let $\mathbf{x} \in \ker(T^\dagger \circ T)$ i.e. $(T^\dagger \circ T)(\mathbf{x}) = \mathbf{0}_V$. Thus, $\langle \mathbf{y}, (T^\dagger \circ T)(\mathbf{x}) \rangle = 0$ for any $\mathbf{y} \in V$. In particular, let $\mathbf{y} = \mathbf{x}$. Then

$$0 = \langle \mathbf{x}, (T^\dagger \circ T)(\mathbf{x}) \rangle = \langle T(\mathbf{x}), T(\mathbf{x}) \rangle = \|T(\mathbf{x})\|_W^2 \implies T(\mathbf{x}) = \mathbf{0}_W \text{ i.e. } \mathbf{x} \in \ker(T).$$

Thus, $\ker(T) = \ker(T^\dagger \circ T)$. ■

The second equality follows by replacing T with $T^\dagger$. Indeed, the equality $\operatorname{rank}(T^\dagger \circ T) = \operatorname{rank}(T)$ becomes

$$\underbrace{\operatorname{rank}((T^\dagger)^\dagger \circ (T^\dagger))}_{=\operatorname{rank}(T \circ T^\dagger)} = \underbrace{\operatorname{rank}(T^\dagger)}_{=\operatorname{rank}(T)}.$$

■

5.0.3 Modulus

Let $T = T^\dagger \geqslant 0$ (i.e. let T be a self-adjoint and positive semidefinite operator). By the spectral theorem, there exists an orthonormal basis $E = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of V consisting of eigenvectors of T. Let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues. These eigenvalues are real since T is self-adjoint, and we can say further that they are non-negative because T is positive semidefinite. Because of this, we may define $S(\mathbf{v}_i) = \sqrt{\lambda_i}\mathbf{v}_i$ and this extends by linearity to all of V since E is an orthonormal basis. The matrix representation of S w.r.t. E is $\operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$ which is clearly Hermitian and so S is self-adjoint. Furthermore, S is positive semidefinite. Also, $S^2 \coloneqq S \circ S$ is given on the basis elements (the eigenvectors of T) by

$$S^2(\mathbf{v}_i) = S(S(\mathbf{v}_i)) = S(\sqrt{\lambda_i}\mathbf{v}_i) = \sqrt{\lambda_i}S(\mathbf{v}_i) = \sqrt{\lambda_i}\sqrt{\lambda_i}\mathbf{v}_i = \lambda\mathbf{v}_i = T(\mathbf{v}_i)$$

and a linear map is uniquely defined by its action on a basis so $S^2 = T$.

S is unique according to a few books/`.pdf` documents I've seen floating about online. I'm yet to convince myself of this fact but leaving this here to include later.

We also denote the relationship $S^2 = T$ by $S = \sqrt{T}$.

Definition 5.0.9 Let $T \colon V \to W$ be a linear map. Since the Hermitian square $T^\dagger \circ T$ of T is a self-adjoint and positive semidefinite operator on V, there exists a unique operator S s.t. $S = \sqrt{T^\dagger \circ T}$. This operator S is called the **modulus of T** and is often denoted by $|T|$.

Remarks 5.0.10

- The relationship $|T|^2 = T^\dagger \circ T$ itself suggests that T may be factored as the modulus $|T|$ of T multiplied by some map that acts as a rotation in analogy with complex numbers and how $z = re^{i\phi}$ where $r = |z|$. This is the subject of **Section 5.1**.
- The modulus $|T|$ and the Hermitian square $T^\dagger \circ T$ of T share the same eigenvectors.
- The name suggests that the modulus of T tells us "how big" the operator T is. Indeed:

$$\begin{aligned}
\||T|(\mathbf{v})\|^2 &= \langle |T|(\mathbf{v}), |T|(\mathbf{v})\rangle \\
&= \langle \mathbf{v}, (|T|)^\dagger(|T|(\mathbf{v}))\rangle \\
&= \langle \mathbf{v}, |T|(|T|(\mathbf{v}))\rangle \quad \text{since } |T| \text{ is self-adjoint} \\
&= \left\langle \mathbf{v}, |T|^2(\mathbf{v}))\right\rangle \\
&= \langle \mathbf{v}, (T^\dagger \circ T)(\mathbf{v}))\rangle \quad \text{since } |T| = \sqrt{T^\dagger \circ T} \\
&= \langle T(\mathbf{v}), T(\mathbf{v})\rangle \\
&= \|T(\mathbf{v})\|^2
\end{aligned}$$

Definition 5.0.11 Eigenvalues of $|T|$ are called the **singular values of T**.

As explained earlier, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $T^\dagger \circ T$, then $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}$ are the eigenvalues of $|T|$, and therefore the singular values of T. We denote these singular values by $s_i := \sqrt{\lambda_i}$.

5.1 Polar Decomposition

Let's diagonalise $T^\dagger \circ T$ to see what happens to these vectors under the action of T. Thanks to the spectral theorem, since $T^\dagger \circ T$ is self-adjoint, there exists a basis $E = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ for V consisting of eigenvectors of $T^\dagger \circ T$ with corresponding real eigenvalues $\lambda_1, \ldots, \lambda_n$.

By the discussion at the beginning of this section, we already know the lengths $\|T(\mathbf{v}_i)\| = \sqrt{\lambda_i}\|\mathbf{v}_i\| = \sigma_i\|\mathbf{v}_i\|$.

How about the geometry of the $T(\mathbf{v}_i)$? Suppose WLOG that $\operatorname{rank}(T) = r$. **Lemma 5.0.7 (b)** tells us that $\operatorname{rank}(T) = \operatorname{rank}(T^\dagger \circ T)$.

Have I mentioned that the rank of a diagonalisable operator is equal to the number of its non-zero eigenvalues? It comes from the part of the Spectral **Theorem 4.3.1** which says that if we denote by $\lambda_1, \ldots, \lambda_k$ the *distinct* eigenvalues of the self-adjoint linear operator $T^\dagger \circ T$, then it's diagonalisable with

$$V = \bigoplus_{i=1}^{k} \ker(T^\dagger \circ T - \lambda_i \operatorname{id}_V).$$

By the Rank-Nullity Theorem on $T^\dagger \circ T$

$$\begin{aligned}
\operatorname{rank}(T^\dagger \circ T) &= -\operatorname{nullity}(T^\dagger \circ T) + \dim(V) \\
&= -\operatorname{nullity}(T^\dagger \circ T) + \dim\left(\bigoplus_{i=1}^{k} \ker(T - \lambda_i \operatorname{id}_V)\right) \\
&= -\operatorname{nullity}(T^\dagger \circ T) + \sum_{i=1}^{k} \dim(\ker(T^\dagger \circ T - \lambda_i \operatorname{id}_V))
\end{aligned}$$

Now notice that

$$\begin{aligned}
\operatorname{nullity}(T^\dagger \circ T) &:= \dim(\ker(T^\dagger \circ T)) \\
&= \dim(\ker(T^\dagger \circ T - 0 \operatorname{id}_V))
\end{aligned}$$

The final piece of the puzzle is from **Theorem 1.8.9**. Since our operator is diagonalisable, its characteristic polynomial trivially splits and so we can say that for each $i = 1, \ldots, k$ the algebraic multiplicity e_i of λ_i is equal to its geometric multiplicity $d_i = \dim(\ker(T^\dagger \circ T - \lambda_i \operatorname{id}_V))$. Finally,

$$\operatorname{rank}(T^\dagger \circ T) = -e_0 + \sum_{i=1}^{k} e_i$$

i.e. that the $\operatorname{rank}(T^\dagger \circ T)$ is equal to its number of non-zero eigenvalues.

It follows that the self-adjoint operator $T^\dagger \circ T$ has exactly r non-zero eigenvalues. We may write the singular values out in such a way that the zeroes all come at the end of the ordered list i.e. $\sigma_i \neq 0$ for $i = 1, \ldots, r$ and $\sigma_i = 0$ for $i = r+1, \ldots, n$. Thus $T(\mathbf{v}_i) = 0$ for $i > r$. Now observe that for $i, j = 1, \ldots, r$

$$\langle T(\mathbf{v}_i), T(\mathbf{v}_j)\rangle_W = \langle \mathbf{v}_i, (T^\dagger \circ T)(\mathbf{v}_j)\rangle_V = \langle \mathbf{v}_i, \lambda_j \mathbf{v}_j\rangle_V = \overline{\lambda_j}\langle \mathbf{v}_i, \mathbf{v}_j\rangle_V = \lambda_j \delta_{ij}.$$

Thus, the $T(\mathbf{v}_i)$ are orthogonal. We can normalise the collection $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_r)\}$ by the assignment

$$\mathbf{v}_i \longmapsto T(\mathbf{v}_i)/\sigma_i =: \mathbf{u}_i$$

to get an orthonormal set $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ in W. For the $T(\mathbf{v}_i)$ with $i > r$, we can simply send them to $\mathbf{0}_W$. Thus, we've defined the linear map $U\colon V \to W$ uniquely by its action

$$U\colon \mathbf{v}_i \longmapsto \begin{cases} \mathbf{u}_i := T(\mathbf{v}_i)/\sigma_i & \text{if } i = 1, \ldots, r \\ \mathbf{0}_W & \text{if } i = r+1, \ldots, n \end{cases}$$

on the basis E for V.

Clearly $\ker(U) = \operatorname{span}(\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n)$, and because E is a basis for V we know that $(\ker U)^\perp = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_r)$, and $V = (\ker U)^\perp \oplus \ker(U)$. On $(\ker U)^\perp$, the map U satisfies

$$\begin{aligned}
\langle U(\mathbf{v}_i), U(\mathbf{v}_j)\rangle_W &= \left\langle \frac{T(\mathbf{v}_i)}{\sigma_i}, \frac{T(\mathbf{v}_j)}{\sigma_j} \right\rangle_W \\
&= \frac{1}{\sigma_i\sigma_j}\langle T(\mathbf{v}_i), T(\mathbf{v}_j)\rangle_W \\
&= \frac{1}{\sigma_i\sigma_j}\lambda_j\langle \mathbf{v}_i, \mathbf{v}_j\rangle_V \\
&= \frac{1}{\sigma_i\sigma_j}\sigma_j^2\delta_{ij} \\
&= \frac{\sigma_j}{\sigma_i}\delta_{ij} \\
&= \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \\
&= \langle \mathbf{v}_i, \mathbf{v}_j\rangle_V
\end{aligned}$$

so U preserves inner products on $(\ker U)^\perp = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_r)$, and sends everything in $\ker(U) = ((\ker U)^\perp)^\perp$ (where equality holds because V is finite-dimensional) to $\mathbf{0}_W$. We give a special name to such a function:

Definition 5.1.1 A linear map $U\colon V \to W$ is called a partial isometry on X if for all $\mathbf{v}, \mathbf{w} \in X$

$$\langle U(\mathbf{v}), U(\mathbf{w})\rangle = \langle \mathbf{v}, \mathbf{w}\rangle,$$

and $U(\mathbf{v}) = \mathbf{0}_W$ for all $\mathbf{v} \in X^\perp$.

Thus, U is partial isometry on $X = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_r) = (\ker U)^\perp$.

This is a tangent that I think may prove fruitful at some point so I am including it here. We already have the direct sum decomposition

$$V = (\ker U)^\perp \oplus \ker(U).$$

It turns out that $U^\dagger \circ U$ is a projection. We can demonstrate this by showing that the direct sum decomposition above is of the form

$$V = \operatorname{image}(U^\dagger \circ U) \oplus \ker(U^\dagger \circ U).$$

Indeed, we already know that $\ker(U) = \ker(U^\dagger \circ U)$ by **Lemma 5.0.8**. What remains to be seen is that $(\ker U)^\perp = \operatorname{image}(U^\dagger \circ U)$.

"$\subseteq$" Let $\mathbf{y} \in \operatorname{image}(U^\dagger \circ U)$ i.e. there exists some $\mathbf{x} \in V$ s.t. $(U^\dagger \circ U)(\mathbf{x}) = \mathbf{y}$. Now take any $\mathbf{z} \in \ker(U)$ and notice that

$$\begin{aligned}
\langle \mathbf{y}, \mathbf{z}\rangle_V &= \left\langle (U^\dagger \circ U)(\mathbf{x}), \mathbf{z} \right\rangle_W \\
&= \langle U(\mathbf{x}), U(\mathbf{z})\rangle_W \\
&= \langle U(\mathbf{x}), \mathbf{0}_W\rangle_W \\
&= 0.
\end{aligned}$$

Therefore, $\mathbf{y} \in (\ker U)^\perp$, and so $\operatorname{image}(U^\dagger \circ U) \subseteq (\ker U)^\perp$.

"$\supseteq$" Take any $\mathbf{v} \in (\ker U)^\perp$. We wish to show that $\mathbf{v} = (U^\dagger \circ U)(\mathbf{x})$ for some $\mathbf{x} \in V$. Since U is a partial isometry on $(\ker U)^\perp$, for any $\mathbf{w} \in (\ker U)^\perp$

$$\begin{aligned} \langle \mathbf{v}, \mathbf{w} \rangle_W &= \langle U(\mathbf{v}), U(\mathbf{w}) \rangle_W \\ &= \langle (U^\dagger \circ U)(\mathbf{v}), \mathbf{w} \rangle_V \quad \text{by the adjoint property} \end{aligned}$$

from which it follows that

$$\Big\langle (U^\dagger \circ U)(\mathbf{v}) - \mathbf{v}, \mathbf{w} \Big\rangle_V = 0 \quad \text{for all } \mathbf{w} \in (\ker U)^\perp.$$

By the inclusion $\operatorname{image}(U^\dagger \circ U) \subseteq (\ker U)^\perp$ just proven, $(U^\dagger \circ U)(\mathbf{v}) \in \operatorname{image}(U^\dagger \circ U) \subseteq (\ker U)^\perp$ and $\mathbf{v}$ is also in $(\ker U)^\perp$ so $(U^\dagger \circ U)(\mathbf{v}) - \mathbf{v} \in (\ker U)^\perp$. Now our equality tells us that $(U^\dagger \circ U)(\mathbf{v}) - \mathbf{v}$ is orthogonal to every element $\mathbf{w}$ of $(\ker U)^\perp$, and is itself an element of $(\ker U)^\perp$. Thus, it is orthogonal to itself i.e. is equal to $\mathbf{0}_V$ i.e. $(U^\dagger \circ U)(\mathbf{v}) = \mathbf{v}$. Thus, $\mathbf{v} \in \operatorname{image}(U^\dagger \circ U)$.

Therefore, $U^\dagger \circ U$ is a projection from V on $\operatorname{image}(U^\dagger \circ U) = (\ker U)^\perp$. We can go one step further and simply notice that $\operatorname{image}(U^\dagger \circ U) = (\ker U)^\perp = (\ker(U^\dagger \circ U))^\perp$ and that these spaces are finite dimensional so this is sufficient for the projection to be orthogonal.

Since $\mathbf{u}_i := T(\mathbf{v}_i)/\sigma_i$, we may equivalently write

$$T(\mathbf{v}_i) = \sigma_i \mathbf{u}_i.$$

This is an important shift in perspective because it facilitates the following manipulation. Indeed, recall that $|T|$ and $T^\dagger \circ T$ share the same eigenvectors and so $|T|(\mathbf{v}_i) = \sigma_i \mathbf{v}_i$. Therefore, we may write

$$\begin{aligned} T(\mathbf{v}_i) &= \sigma_i \mathbf{u}_i \\ &= \sigma_i U(\mathbf{v}_i) \\ &= U(\sigma_i \mathbf{v}_i) \\ &= U(|T|(\mathbf{v}_i)) \\ &= (U \circ |T|)(\mathbf{v}_i) \end{aligned}$$

i.e. the action of T on each eigenvector $\mathbf{v}_i$ of $T^\dagger \circ T$ is the result of scaling each eigenvector by the singular value (with $|T|$) and then performing a partial isometry on the resulting vector.

Definition 5.1.2 For a linear map $T\colon V \to W$ between finite-dimensional inner product spaces, the decomposition

$$T = U \circ |T|$$

where $|T| := \sqrt{T^\dagger \circ T}\colon V \to V$ is the modulus of T, and $U\colon V \to W$ is a partial isometry on $(\ker U)^\perp$, is called a **polar[a] decomposition of T**.

[a] The name draws from the analogous fact about a complex number z admitting the polar representation $z = re^{i\phi}$.

5.2 Singular Value Decomposition

Starting with a polar decomposition of T, we now express any $\mathbf{x} \in V$ by the unique linear combination

$$\mathbf{x} = \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i$$

and then offer an expression for the action of T on any such $\mathbf{x} \in V$:

$$
\begin{aligned}
T(\mathbf{x}) &= T\left(\sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i\right) \\
&= \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle T(\mathbf{v}_i) \\
&= \sum_{i=1}^{r} \langle \mathbf{x}, \mathbf{v}_i \rangle T(\mathbf{v}_i) \quad \text{since } \operatorname{rank}(T) = r \\
&= \sum_{i=1}^{r} \langle \mathbf{x}, \mathbf{v}_i \rangle (U \circ |T|)(\mathbf{v}_i) \\
&= \sum_{i=1}^{r} \langle \mathbf{x}, \mathbf{v}_i \rangle \sigma_i \mathbf{u}_i \\
&= \sum_{i=1}^{r} \sigma_i \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{u}_i
\end{aligned}
$$

This is a **singular value decomposition of T**. We state it as a theorem.

Theorem 5.2.1 (SVD) Let $T\colon V \to W$ be a linear map of rank r, and denote by $\sigma_1, \ldots, \sigma_r$ the non-zero singular values of T. Then there exist orthonormal collections $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq V$ and $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\} \subseteq W$ s.t. for every $\mathbf{x} \in V$:

$$T(\mathbf{x}) = \sum_{i=1}^{r} \sigma_i \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{u}_i.$$

In the development of our theory, we had an orthonormal basis $E = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of V (where $\dim(V) = n$) consisting of eigenvectors (corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$) of $T^\dagger \circ T$ (and $|T|$ but with singular values $\sigma_1, \ldots, \sigma_n$ as the eigenvalues). We constructed the orthonormal collection $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ where $r = \operatorname{rank}(T) = \operatorname{rank}(T^\dagger \circ T)$.

Now we extend the orthonormal collection $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\} \subseteq W$ to an orthonormal basis $F = (\mathbf{u}_1, \ldots, \mathbf{u}_m)$ for W, where $m = \dim(W)$.

From the singular value decomposition, we observe that for $j = 1, \ldots, r$

$$
\begin{aligned}
T(\mathbf{v}_j) &= \sum_{i=1}^{r} \sigma_i \langle \mathbf{v}_j, \mathbf{v}_i \rangle \mathbf{u}_i \\
&= \sigma_j \mathbf{u}_j \quad \text{by the orthonormality of } E,
\end{aligned}
$$

and for $j > r$ we have $T(\mathbf{v}_j) = \mathbf{0}_W$. Therefore, the matrix $[F, T, E]$ of T with respect to the bases E of V and F of W, historically denoted by Σ, is given by:

$$\Sigma_{ij} = \begin{cases} \sigma_i & \text{if } 1 \leqslant i = j \leqslant r, \\ 0 & \text{otherwise.} \end{cases}$$

Now we denote by E_{std} and F_{std} the standard bases of $\mathbb{K}^{n,1}$ and $\mathbb{K}^{m,1}$ respectively.

$$
\begin{tikzcd}
\mathbb{K}^{n,1} \arrow[rrr, "{\left[T\right]_{E}^{F} = (\underline{\mathbf{x}} \longmapsto [F,T,E]\underline{\mathbf{x}})}"] \arrow[dr, "\Phi_E"] & & & \mathbb{K}^{m,1} \arrow[dl, "\Psi_F"'] \\
& V \arrow[r, "T"] & W & \\
& V \arrow[u, "\mathrm{id}_V"] \arrow[r, "T"'] & W \arrow[u, "\mathrm{id}_W"'] & \\
\mathbb{K}^{n,1} \arrow[uuu, "{\left[\mathrm{id}_V\right]_{E_{\mathrm{std}}}^{E}}"] \arrow[ur, "\Phi_{E_{\mathrm{std}}}"'] \arrow[rrr, "{\left[T\right]_{E_{\mathrm{std}}}^{F_{\mathrm{std}}} = (\underline{\mathbf{x}} \longmapsto [F_{\mathrm{std}},T,E_{\mathrm{std}}]\underline{\mathbf{x}})}"'] & & & \mathbb{K}^{m,1} \arrow[uuu, "{\left[\mathrm{id}_W\right]_{F_{\mathrm{std}}}^{F}}"'] \arrow[ul, "\Psi_{F_{\mathrm{std}}}"]
\end{tikzcd}
$$

By **Theorem 1.5.3**, recall that

$$\left[T\right]_{E}^{F} = \left[\mathrm{id}_W\right]_{F_{\mathrm{std}}}^{F} \circ \left[T\right]_{E_{\mathrm{std}}}^{F_{\mathrm{std}}} \circ \left(\left[\mathrm{id}_V\right]_{E_{\mathrm{std}}}^{E}\right)^{-1}.$$

Denote by

- $P := [E, \mathrm{id}_V, E_{\mathrm{std}}]$, which is the $n \times n$ matrix whose columns are the orthonormal basis vectors $\mathbf{v}_i$ of E for V, and
- $Q := [F, \mathrm{id}_W, F_{\mathrm{std}}]$ which is the $m \times m$ matrix whose columns are the orthonormal basis vectors $\mathbf{u}_i$ of F for W.

It follows that P and Q are unitary matrices by **Proposition 3.3.13**. Finally, we can write down our expression for the matrix representation of T with respect to the standard bases.

$$
\begin{aligned}
[F_{\mathrm{std}}, T, E_{\mathrm{std}}] &= Q[F, T, E]P^{-1} \\
&= Q[F, T, E]\overline{P}^{\top} \\
&= \begin{bmatrix} | & & | \\ \mathbf{u}_1 & \cdots & \mathbf{u}_m \\ | & & | \end{bmatrix} \begin{bmatrix} \mathrm{diag}(\sigma_1, \ldots, \sigma_r) & 0_{r,\, n-r} \\ 0_{m-r,\, r} & 0_{m-r,\, n-r} \end{bmatrix} \begin{bmatrix} — & \overline{\mathbf{v}_1}^{\top} & — \\ & \vdots & \\ — & \overline{\mathbf{v}_n}^{\top} & — \end{bmatrix}
\end{aligned}
$$

We collect the above discussion in a theorem too:

Theorem 5.2.2 (SVD Matrix Representation) Let A be an $m \times n$ matrix of rank r with positive singular values $\sigma_1, \ldots, \sigma_r$, and denote by Σ the matrix defined by

$$\Sigma_{ij} = \begin{cases} \sigma_i & \text{if } 1 \leqslant i = j \leqslant r, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exist unitary matrices Q and P such that $A = Q\Sigma P^{-1}$.

Since P is unitary, this theorem is also sometimes stated as $A = Q\Sigma\overline{P}^{\top}$. Either works because $P^{-1} = \overline{P}^{\top}$.

There are some applications of SVD that I'd like to investigate at some point.

- ~~Principal axes of the ellipsoid which is the image of the unit ball~~ This turned out to be a very immediate consequence of the theory!
- The condition number of a matrix (the ratio σ_1/σ_n of the largest σ_1 and smallest σ_n singular values)
- Moore-Penrose (pseudo)inverse

The SVD of a matrix $A \in \mathbb{K}^{m,n}$ representing a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ (with respect to the standard orthonormal bases $E_{\text{std}} = (\mathfrak{e}_1, \dots, \mathfrak{e}_n)$ of $\mathbb{K}^{n,1}$, and $F_{\text{std}} = (\mathfrak{f}_1, \dots, \mathfrak{f}_m)$ of $\mathbb{K}^{m,1}$) is the following composition of transformations:

- Begin with a unitary transformation $\overline{P}^{\top} = P^{-1}$ that rotates/reflects the standard basis E_{std} of $\mathbb{K}^{n,1}$ into the eigenbasis[1] $E = ([E_{\text{std}}, \mathbf{v}_1], \dots, [E_{\text{std}}, \mathbf{v}_n])$ of $\mathbb{K}^{n,1}$ consisting of eigenvectors of $T^\dagger \circ T$. (Equivalently, E is also the eigenbasis of $|T|$, and this lends itself to the scaling in the next bullet point.)

  - The eigenvectors $\mathbf{v}_i$ of $T^\dagger \circ T$ are called the **right singular vectors of** T.
  - When it comes to matrices, the **right singular vectors of** A are the eigenvectors $[E_{\text{std}}, \mathbf{v}_i]$ of $\overline{A}^{\top} A$.
  - If we instead wish to speak of $T \circ T^\dagger$, then replace all instances of 'right' with 'left'.

  The columns of P are the right singular vectors of A.

- Then we apply the matrix Σ. This has varying shape depending on the dimensions of the domain and codomain of the associated linear transformation:

  - If $m < n$, then Σ is the composition of a "dimension reducer" and a diagonal matrix.
  - If $m > n$, then Σ is the composition of a "dimension extender" (which appends $n - m$ zero components to our vectors) and a diagonal matrix.
  - If $m = n$, then Σ is simply a diagonal matrix.

  In all of these cases, the diagonal matrix acts on $\mathbb{K}^{n,1}$ by scaling (with scale factors $\sigma_i > 0$ for $i = 1, \dots, r$ and $\sigma_i = 0$ for $i > r$) the axes $[E_{\text{std}}, \mathbf{v}_i]$ of the eigenbasis E.

- Finally (we extend to a basis F of the codomain $\mathbb{K}^{m,1}$ of the linear transformation and so) Q is the matrix which rotates the basis $F = ([F_{\text{std}}, \mathbf{u}_1], \dots, [F_{\text{std}}, \mathbf{u}_m])$ back to F_{std}.

Remarks 5.2.3

- The following two examples will highlight what kind of action Σ has and were taken from fantastic YouTube video called **SVD Visualized** by a channel named `@visualkernel`. We say that Σ acts by reducing dimensions if it behaves similarly to the following example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} x \\ y \end{bmatrix},$$

  and by Σ extending dimensions, it can be thought of as something like

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

[1] It might seem pedantic to assert that these vectors are with respect to the standard basis E_{std} of $\mathbb{K}^{n,1}$.

- I think it's most instructive to discuss with an actual example how the SVD of a matrix offers a concrete form for its colspace and nullspace (and their complements) as the linear spans of certain subsets of the columns of P and Q. See more in **Section 6.2.1**.

Example 5.2.4 Let's apply this to the closed unit ball $\overline{\mathbb{B}_1(\mathbf{0})} \subseteq \mathbb{R}^{n,1}$. Suppose that $A \in \mathbb{R}^{m,n}$ is the matrix of a linear transformation $T\colon \mathbb{R}^{n,1} \to \mathbb{R}^{m,1}$ of rank $r \leqslant \min(m, n)$. By the orthogonality of P, the matrix $P^\top$ simply rotates/reflects the unit ball so that the standard basis vectors are now the eigenbasis of $A^\top A$. Then the matrix Σ scales in the direction of the eigenbasis axes. Under Σ, the image $\mathbf{y}$ of $\mathbf{x}$ is given by

$$\begin{bmatrix} y_1 \\ \vdots \\ y_r \\ y_{r+1} \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} \operatorname{diag}(\sigma_1, \dots, \sigma_r) & 0_{r,\, n-r} \\ 0_{m-r,\, r} & 0_{m-r,\, n-r} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sigma_1 x_1 \\ \vdots \\ \sigma_r x_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

i.e. $x_i = y_i/\sigma_i$ for $i = 1, \dots, r$, and $y_i = 0$ for $i > r$. Now observe that

$$1 \geqslant \|\mathbf{x}\|^2 = \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{r} \left(\frac{y_i}{\sigma_i}\right)^2$$

which is the inequality that characterises an r-dimensional ellipsoid with principal semi-axes of lengths $\sigma_1, \dots, \sigma_r$. ▶

CHAPTER 6

Linking Back to Machine Learning

This links back to 102.

6.1 Ordinary Least-Squares and Orthogonal Projection

Recall that the design matrix associated with the training data $\mathcal{T} = \{(x^{(i)}, y^{(i)})\}_{i=1,\dots,n}$ collected for a linear regression (supervised learning) problem is the $n \times (p+1)$ real matrix of feature vectors augmented by a zeroth column of 1's (to account for an intercept/bias term):

$$\mathcal{X} = \begin{bmatrix} — & (x^{(1)})^\top & — \\ & \vdots & \\ — & (x^{(n)})^\top & — \end{bmatrix} = \begin{bmatrix} 1 & x_1^{(1)} & \cdots & x_p^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_1^{(n)} & \cdots & x_p^{(n)} \end{bmatrix}.$$

The ordinary least-squares problem is to find parameters $\boldsymbol{\theta} \in \mathbb{R}^{p+1,1}$ that minimise our loss function

$$J(\boldsymbol{\theta}) = \|\mathcal{X}\boldsymbol{\theta} - \mathbf{y}\|^2, \qquad \text{where } \mathbf{y} = \begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(n)} \end{bmatrix} \in \mathbb{R}^{n,1}.$$

In other words, the problem is

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{p+1,1}} \|\mathcal{X}\boldsymbol{\theta} - \mathbf{y}\|^2.$$

Since the norm is non-negative, and $f(x) = x^2$ is monotonically increasing (and hence order-preserving) over $[0, +\infty)$, this problem is equivalent to

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{p+1,1}} \|\mathcal{X}\boldsymbol{\theta} - \mathbf{y}\|.$$

Since $\boldsymbol{\theta}$ runs over $\mathbb{R}^{p+1,1}$, $\mathcal{X}\theta =: \mathbf{w}$ runs over colspace$(\mathcal{X})$ and we may reframe the minimisation problem as

$$\min_{\mathbf{w} \in \operatorname{colspace}(\mathcal{X}) \subseteq \mathbb{R}^{n,1}} \|\mathbf{w} - \mathbf{y}\|.$$

Recall the discussion that led to **Corollary 2.4.21**. This means that the unique minimiser of $\|\mathbf{w} - \mathbf{y}\|$ over colspace$(\mathcal{X})$ is the orthogonal projection of $\mathbf{y}$ onto the image of $\mathcal{X}$, denoted by $\mathbf{u} = P_{\operatorname{colspace}(\mathcal{X})}(\mathbf{y})$. The corresponding set of minimisers in $\mathbb{R}^{p+1,1}$ to $\mathbf{u}$ is the collection

$$\{\boldsymbol{\theta} \in \mathbb{R}^{p+1,1} : \mathcal{X}\boldsymbol{\theta} = \mathbf{u}\}.$$

Since $\mathbf{u}$ is the unique vector in colspace$(\mathcal{X})$ satisfying $\mathbf{y} - \mathbf{u} \in (\operatorname{colspace} \mathcal{X})^\perp$ i.e. for any $\varphi \in \mathbb{R}^{p+1,1}$

$$\begin{aligned} 0 &= \langle \mathbf{y} - \mathbf{u}, \mathcal{X}\varphi \rangle_{\mathbb{R}^{n,1}} \\ &= \left\langle \overline{\mathcal{X}}^\top (\mathbf{y} - \mathbf{u}), \varphi \right\rangle_{\mathbb{R}^{n,1}} \quad \text{by the adjoint property} \\ &= \left\langle \mathcal{X}^\top (\mathbf{y} - \mathbf{u}), \varphi \right\rangle_{\mathbb{R}^{n,1}} \end{aligned}$$

which implies, by **Lemma 2.4.3**, that $0_{\mathbb{R}^{n,1}} = \mathcal{X}^\top(\mathbf{y} - \mathbf{u})$ i.e. $\mathcal{X}^\top\mathbf{y} = \mathcal{X}^\top\mathbf{u}$. Any $\hat{\boldsymbol{\theta}} \in \{\boldsymbol{\theta} \in \mathbb{R}^{p+1,1} : \mathcal{X}\boldsymbol{\theta} = \mathbf{u}\}$ then satisfies what are known as the **normal equations**

$$\mathcal{X}^\top\mathbf{y} = \mathcal{X}^\top\mathcal{X}\hat{\boldsymbol{\theta}},$$

and any such $\hat{\boldsymbol{\theta}}$ is called a **least-squares estimate**.

6.2 Ordinary Least-Squares via SVD

Typically, the design matrix $\mathcal{X}$ is very tall (and possibly doesn't have full rank). We can still decompose such a matrix via singular value decomposition:

$$\begin{aligned}\mathcal{X} &= Q\Sigma P^{-1} \\ &= Q\Sigma\overline{P}^{\top} \\ &= Q\Sigma P^{\top} \quad \text{since } \mathcal{X} \text{ is a real matrix,}\end{aligned}$$

where $\mathcal{X} \in \mathbb{R}^{n,p+1}$, $P \in \mathbb{R}^{p+1,p+1}$, and $Q \in \mathbb{R}^{n,n}$. This decomposition is a means to cast the ordinary least-squares optimisation problem into a different geometry that makes the problem simpler in a sense. Observe that

$$\begin{aligned}\|\mathcal{X}\boldsymbol{\theta} - \mathbf{y}\|^2 &= \left\|Q\Sigma P^{\top}\boldsymbol{\theta} - I_n\mathbf{y}\right\|^2 \\ &= \left\|Q\Sigma P^{\top}\boldsymbol{\theta} - QQ^{\top}\mathbf{y}\right\|^2 \\ &= \left\|Q(\Sigma P^{\top}\boldsymbol{\theta} - Q^{\top}\mathbf{y})\right\|^2 \\ &= \left\|\Sigma P^{\top}\boldsymbol{\theta} - Q^{\top}\mathbf{y}\right\|^2 \quad \text{since } Q \text{ is orthogonal} \\ &= \|\Sigma\mathbf{z} - \mathbf{c}\|^2\end{aligned}$$

where $\mathbf{z} := P^{\top}\boldsymbol{\theta}$ and $\mathbf{c} := Q^{\top}\mathbf{y}$. Thus, we cast

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^{p+1,1}} \|\mathcal{X}\boldsymbol{\theta} - \mathbf{y}\|^2 \quad \overset{\text{into}}{\rightsquigarrow} \quad \min_{\mathbf{z} \in \mathbb{R}^{p+1,1}} \|\Sigma\mathbf{z} - \mathbf{c}\|^2$$

Now note that

$$\Sigma\mathbf{z} = \Sigma\begin{bmatrix} z_1 \\ \vdots \\ z_r \\ z_{r+1} \\ \vdots \\ z_{p+1}\end{bmatrix} = \begin{bmatrix}\sigma_1 z_1 \\ \vdots \\ \sigma_r z_r \\ 0 \\ \vdots \\ 0\end{bmatrix} \in \mathbb{R}^{n,1}$$

and so the norm has a simpler expression

$$\|\Sigma\mathbf{z} - \mathbf{c}\|^2 = \sum_{i=1}^{r}(\sigma_i z_i - c_i)^2 + \underbrace{\sum_{i=r+1}^{n} c_i^2}_{\text{constant}}.$$

Thus, we need only minimise the first sum which gives $z_i = c_i/\sigma_i$ for $i = 1, \ldots, r$ and the remaining $z_{r+1}, \ldots, z_{p+1}$ are free to be chosen. Therefore, the collection of minimisers of this recast minimisation problem is

$$\left\{\mathbf{z} \in \mathbb{R}^{p+1,1} : \mathbf{z} = \begin{bmatrix} c_1/\sigma_1 \\ \vdots \\ c_r/\sigma_r \\ * \\ \vdots \\ * \end{bmatrix} + \mathbf{z}' \text{ where } \mathbf{z}' \in \operatorname{nullspace}(\Sigma)\right\}$$

and we can pull these minimisers back to the minimisation problem over $\boldsymbol{\theta}$:

$$\left\{ \boldsymbol{\theta} \in \mathbb{R}^{p+1,1} : \boldsymbol{\theta} = P\mathbf{z} = P \begin{bmatrix} c_1/\sigma_1 \\ \vdots \\ c_r/\sigma_r \\ * \\ \vdots \\ * \end{bmatrix} + P\mathbf{z}' \right\} \xleftarrow[\mathbf{z}=P^\top \boldsymbol{\theta} \iff P\mathbf{z}=\boldsymbol{\theta}]{\text{cast back via}} \{\mathbf{z} \in \mathbb{R}^{p+1,1} : \ldots\}$$

Let's investigate these pre-images $\boldsymbol{\theta}$ of the minimisers. Observe that

$$\mathcal{X}\boldsymbol{\theta} = \mathcal{X} \left(P \begin{bmatrix} c_1/\sigma_1 \\ \vdots \\ c_r/\sigma_r \\ * \\ \vdots \\ * \end{bmatrix} + P\mathbf{z}' \right) = \mathcal{X}P \begin{bmatrix} c_1/\sigma_1 \\ \vdots \\ c_r/\sigma_r \\ * \\ \vdots \\ * \end{bmatrix} + \underbrace{\mathcal{X}P\mathbf{z}'}$$

The under-braced term is

$$\begin{aligned} \mathcal{X}P\mathbf{z}' &= Q\Sigma P^\top P\mathbf{z}' \\ &= Q\Sigma I_{p+1}\mathbf{z}' \\ &= Q\Sigma\mathbf{z}' \\ &= Q\mathbf{0}_{\mathbb{R}^{n,1}} \quad \text{since } \mathbf{z}' \in \text{nullspace}(\Sigma) \\ &= \mathbf{0}_{\mathbb{R}^{n,1}} \end{aligned}$$

i.e. $\mathbf{z}' \in \text{nullspace}(\Sigma) \implies P\mathbf{z}' \in \text{nullspace}(\mathcal{X})$. In fact, the reverse inclusion also holds because P is invertible. This means that

$$\boldsymbol{\theta} = P \begin{bmatrix} c_1/\sigma_1 \\ \vdots \\ c_r/\sigma_r \\ * \\ \vdots \\ * \end{bmatrix} + \boldsymbol{\varphi}' \quad \text{where } \boldsymbol{\varphi}' \in \text{nullspace}(\mathcal{X})$$

i.e. the minimisers of the original problem over $\boldsymbol{\theta}$ differ by $\boldsymbol{\varphi} \in \text{nullspace}(\mathcal{X})$. Thus, we see that any such minimiser is acted on by $\mathcal{X}$ as follows

$$\mathcal{X}\boldsymbol{\theta} = Q\Sigma P^\top P \begin{bmatrix} c_1/\sigma_1 \\ \vdots \\ c_r/\sigma_r \\ * \\ \vdots \\ * \end{bmatrix} = Q\Sigma \begin{bmatrix} c_1/\sigma_1 \\ \vdots \\ c_r/\sigma_r \\ * \\ \vdots \\ * \end{bmatrix} = Q \begin{bmatrix} c_1 \\ \vdots \\ c_r \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sum_{i=1}^{r} c_i \mathbf{u}_i$$

and Σ sends the final $(p+1) - r$ coordinates to zero[1], so their choice is truly arbitrary in the eyes of colspace($\mathcal{X}$) i.e. any such minimiser's $z_{r+1}, \ldots, z_{p+1}$ doesn't matter, for they all give the same image vector under $\mathcal{X}$.

[1]Recall that depending on whether n is equal to, less than, or greater than $p+1$, Σ will either not change the dimensionality of the vector, or extending it, or cut it short. We typically assume that n is greater than $p+1$, so this is an augmentation of the vector.

The canonical choice is to pick them in a way that minimises the norm of $\boldsymbol{\theta}$. How do we do this? Since P is orthogonal, $\|P\mathbf{x}\| = \|\mathbf{x}\|$ for every $\mathbf{x} \in \mathbb{R}^{p+1,1}$. Thus

$$\|\mathbf{z}\| = \left\|P^\top \boldsymbol{\theta}\right\| = \left\|PP^\top \boldsymbol{\theta}\right\| = \|\boldsymbol{\theta}\|,$$

and we set all the $z_{r+1}, \ldots, z_{p+1}$ to zero.

Definition 6.2.1 Recall that $\Sigma = \begin{bmatrix} \operatorname{diag}(\sigma_1, \ldots, \sigma_r) & 0_{r,\,(p+1)-r} \\ 0_{n-r,\,r} & 0_{n-r,\,(p+1)-r} \end{bmatrix}$.

Some books give an explicit name to the matrix that acts on vectors in $\mathbb{R}^{p+1,\,1}$ in the following way:

$$\Sigma^{\sim} \begin{bmatrix} c_1 \\ \vdots \\ c_r \\ c_{r+1} \\ \vdots \\ c_{p+1} \end{bmatrix} = \begin{bmatrix} c_1/\sigma_1 \\ \vdots \\ c_r/\sigma_r \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n,1}.$$

We call

$$\Sigma^{\sim} := \begin{bmatrix} \operatorname{diag}(1/\sigma_1, \ldots, 1/\sigma_r) & 0_{r,\,(p+1)-r} \\ 0_{n-r,\,r} & 0_{n-r,\,(p+1)-r} \end{bmatrix}$$

the **pseudo-inverse of Σ**.

Observe that the output vector is the form of the vector we arrived at through the canonical minimum-norm reasoning, and so we may write our minimum-norm minimisers as

$$\boldsymbol{\theta} = P \begin{bmatrix} c_1/\sigma_1 \\ \vdots \\ c_r/\sigma_r \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \boldsymbol{\varphi}' = P\Sigma^{\sim}\mathbf{c} + \boldsymbol{\varphi}' = P\Sigma^{\sim}Q^\top \mathbf{y} + \boldsymbol{\varphi}'$$

where $\boldsymbol{\varphi}' \in \operatorname{nullspace}(\mathcal{X})$. As it stands, there's really no explicit way one can compute $\boldsymbol{\varphi}'$ with the SVD. The goal of this section is to arrive at a solution that's totally computable from the SVD. **We'll revisit this shortly**.

We now compute the c_i explicitly. Recall that

$$\mathbf{c} := Q^\top \mathbf{y} = \begin{bmatrix} — & \mathbf{u}_1^\top & — \\ & \vdots & \\ — & \mathbf{u}_n^\top & — \end{bmatrix} \mathbf{y} = \begin{bmatrix} \mathbf{u}_1^\top \mathbf{y} \\ \vdots \\ \mathbf{u}_n^\top \mathbf{y} \end{bmatrix}$$

and so $c_i = \mathbf{u}_i^\top \mathbf{y} = \langle \mathbf{y}, \mathbf{u}_i \rangle$.

$$\therefore\ \mathcal{X}\boldsymbol{\theta} = \sum_{i=1}^{r} \langle \mathbf{y}, \mathbf{u}_i \rangle \mathbf{u}_i$$

which matches precisely the orthogonal projection of $\mathbf{y}$ onto the subspace of $\mathbb{R}^n$ spanned by $\mathbf{u}_1, \ldots, \mathbf{u}_r$. All that remains to observe is that this subspace is indeed $\operatorname{colspace}(\mathcal{X})$. Indeed, the SVD of $\mathcal{X}$ offers exactly the form $\operatorname{colspace}(\mathcal{X})$ takes. I'll also remark on $\operatorname{nullspace}(\mathcal{X})$ as promised in **Remarks 5.2.3**. In fact, the following subsection offers useful information for our other **open lead**.

6.2.1 SVD tells us about colspace and nullspace

- Since P is an orthogonal matrix, so too is $P^\top = P^{-1}$ i.e. its associated linear map is surjective i.e. its image is all of $\mathbb{R}^{p+1}$. Therefore,

$$\operatorname{colspace}(\mathcal{X}) = \left\{Q\Sigma\mathbf{z} : \mathbf{z} \in \mathbb{R}^{p+1}\right\}.$$

Now observe that Σ annihilates all but the first r coordinates of $\mathbf{z}$ (those first r are scaled according to the singular values σ_i) i.e. $\Sigma\mathbf{z} \in \operatorname{span}(\{\mathfrak{e}_1, \ldots, \mathfrak{e}_r\})$. Finally, applying Q tells us that $Q\Sigma\mathbf{z} \in \operatorname{span}(\{Q\mathfrak{e}_1, \ldots, Q\mathfrak{e}_r\})$ so $\operatorname{colspace}(\mathcal{X}) \subseteq \operatorname{span}(\{\mathbf{u}_1, \ldots, \mathbf{u}_r\})$. Conversely, $\mathcal{X}(P\mathfrak{e}_i) = Q\Sigma\mathfrak{e}_i = \sigma_i\mathbf{u}_i$ so $\mathbf{u}_i \in \operatorname{image}(\mathcal{X})$ for $i = 1, \ldots, r$.

$$\therefore\ \operatorname{colspace}(\mathcal{X}) = \operatorname{span}(\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}).$$

We wrap this up nicely into the statement that the columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of Q satisfy

  - $(\mathbf{u}_1, \ldots, \mathbf{u}_r)$ being an orthonormal basis for $\operatorname{colspace}(\mathcal{X})$, and
  - $(\mathbf{u}_{r+1}, \ldots, \mathbf{u}_n)$ being an orthonormal basis for $(\operatorname{colspace}\mathcal{X})^\top$.

- For the nullspace of $\mathcal{X}$, observe that

$$\mathcal{X}\boldsymbol{\theta} = \mathbf{0}_{\mathbb{R}^{n,1}} \iff Q\Sigma P^\top\boldsymbol{\theta} = \mathbf{0}_{\mathbb{R}^{n,1}} \iff \Sigma P^\top\boldsymbol{\theta} = \mathbf{0}_{\mathbb{R}^{n,1}}.$$

By the same logic as before, P is orthogonal and so we may parameterise $P^\top\boldsymbol{\theta}$ by $\mathbf{c}$ which varies over $\mathbb{R}^{p+1,1}$. The condition $\Sigma\mathbf{c} = \mathbf{0}_{\mathbb{R}^{n,1}}$ forces $c_1 = \ldots = c_r = 0$, while the $c_{r+1}, \ldots, c_{p+1}$ are free to vary. Thus, $P^\top\boldsymbol{\theta} = \mathbf{c} \in \operatorname{span}(\mathfrak{e}_{r+1}, \ldots, \mathfrak{e}_{p+1})$. This is equivalent to $\boldsymbol{\theta} = P\mathbf{c} = c_{r+1}\mathbf{v}_{r+1} + \ldots + c_{p+1}\mathbf{v}_{p+1}$ i.e. $\boldsymbol{\theta} \in \operatorname{span}(\mathbf{v}_{r+1}, \ldots, \mathbf{v}_{p+1})$.

$$\therefore\ \operatorname{nullspace}(\mathcal{X}) = \operatorname{span}(\mathbf{v}_{r+1}, \ldots, \mathbf{v}_{p+1}).$$

- Since $(\mathbf{v}_1, \ldots, \mathbf{v}_{p+1})$ is an orthonormal basis for $\mathbb{R}^{p+1,1}$ (namely, the columns of P), the orthogonal complement of the nullspace is simply

$$(\operatorname{nullspace}\mathcal{X})^\top = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_r).$$

- For any $\mathbf{z} \in \mathbb{R}^{n,1}$,

$$\mathcal{X}^\top\mathbf{z} = (Q\Sigma P^\top)\mathbf{z} = P\Sigma^\top Q^\top\mathbf{z} = P\Sigma^\top\mathbf{c}$$

where $Q^\top\mathbf{z} = \mathbf{c}$ varies over $\mathbb{R}^{n,1}$. Thus, $\Sigma^\top\mathbf{c}$ varies over $\operatorname{colspace}(\Sigma^\top) = \operatorname{span}(\mathfrak{e}_1, \ldots, \mathfrak{e}_r) \subseteq \mathbb{R}^{p+1,1}$. Applying P sends each $\mathfrak{e}_i$ to $\mathbf{v}_i$ for $i = 1, \ldots, r$. Thus,

$$\begin{aligned}\operatorname{colspace}(\mathcal{X}^\top) &= \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_r) \\ &= (\operatorname{nullspace}\mathcal{X})^\top.\end{aligned}$$

6.3 Closing Remarks

Conclusion 1

The SVD of $\mathcal{X}$ provides an explicit formula for the projection that is simply truncating the expansion of $\mathbf{y}$ with respect to the orthonormal basis $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ to the first r terms i.e. SVD is really a decomposition that rotates into an orthonormal coordinate system for which orthogonal projection is just truncation.

This was quite a geometric perspective. We could equally have taken the normal equations, and followed the algebra through the SVD to obtain these coefficients c_i.

That would've involved a calculation like this to get started:

$$\begin{aligned}\mathcal{X}^\top\mathcal{X} &= (Q\Sigma P^\top)^\top(Q\Sigma P^\top)\\ &= P\Sigma^\top Q^\top Q\Sigma P^\top\\ &= P\Sigma^\top I_n\Sigma P^\top\\ &= P\Sigma^\top\Sigma P^\top\end{aligned}$$

The matrix $\Sigma^\top\Sigma$ is a square matrix with dimension $(p+1)\times(p+1)$. We can work out the $(i,j)^{\text{th}}$ entry:

$$\begin{aligned}(\Sigma^\top\Sigma)_{ij} &= \langle i^{\text{th}} \text{ row of } \Sigma^\top, j^{\text{th}} \text{ column of } \Sigma\rangle\\ &= \left\langle i^{\text{th}} \text{ column of } \Sigma, j^{\text{th}} \text{ column of } \Sigma\right\rangle.\end{aligned}$$

For $i=1,\ldots,r$, the only non-zero entry of the i^{th} column of Σ is in the i^{th} place.

- Let $i,j>r$. The inner product is clearly 0 so $(\Sigma^\top\Sigma)_{ij}=0$ for $i,j>r$.
- Let $i,j=1,\ldots,r$.
  - If $i\neq j$, then $(\Sigma^\top\Sigma)_{ij}=0$.
  - Else $i=j$, and so $(\Sigma^\top\Sigma)_{ii}=\sigma_i^2=\lambda_i$, where λ_i are the eigenvalues of $\mathcal{X}^\top\mathcal{X}$.

Therefore, $\Sigma^\top\Sigma=\operatorname{diag}(\sigma_1^2,\ldots,\sigma_r^2,\underbrace{0,\ldots,0}_{(p+1)-r})$ and so

$$\mathcal{X}^\top\mathcal{X} = P\ \operatorname{diag}(\sigma_1^2,\ldots,\sigma_r^2,0,\ldots,0)\ P^\top$$

This actually leads us to the Gram matrix being orthogonally diagonalisable which is a nice by-product.

Conclusion 2

We may re-write φ' from our minimum-norm general solution

$$P\Sigma^{\sim}Q^\top\mathbf{y}+\boldsymbol{\varphi}'$$

by considering a suitable orthogonal decomposition of $\mathbb{R}^{p+1,1}$. We want one of the components of this decomposition to be nullspace$(\mathcal{X})$. Consider the orthogonal projection of $\mathbb{R}^{p+1,1}$ onto nullspace$(\mathcal{X})$. Let $P_{\text{nullspace}(\mathcal{X})}\colon\mathbb{R}^{p+1,1}\to\mathbb{R}^{p+1,1}$ denote the orthogonal projection of $\mathbb{R}^{p+1,1}$ onto its image$(T)=$ nullspace$(\mathcal{X})$. Note that

$$\begin{aligned}\mathbb{R}^{p+1,1} &= \operatorname{image}(P_{\text{nullspace}(\mathcal{X})})\oplus\ker(P_{\text{nullspace}(\mathcal{X})})\\ &= \operatorname{image}(P_{\text{nullspace}(\mathcal{X})})\oplus(\operatorname{image}(P_{\text{nullspace}(\mathcal{X})}))^\perp \quad \text{since the projection is orthogonal}\\ &= \operatorname{nullspace}(\mathcal{X})\oplus(\operatorname{nullspace}\mathcal{X})^\top\\ &= \operatorname{nullspace}(\mathcal{X})\oplus\operatorname{colspace}(\mathcal{X}^\top)\end{aligned}$$

Thus, we may write our basis for $\mathbb{R}^{p+1,1}$ as

$$E=(\underbrace{\mathbf{v}_{r+1},\ldots,\mathbf{v}_{p+1}}_{\text{nullspace}(\mathcal{X})},\ \underbrace{\mathbf{v}_1,\ldots,\mathbf{v}_r}_{\text{colspace}(\mathcal{X}^\top)})$$

and so the matrix representation of $P_{\text{nullspace}(\mathcal{X})}$ with respect to E is

$$[E,P_{\text{nullspace}(\mathcal{X})},E]=\begin{bmatrix}I_{(p+1)-r} & 0\\ 0 & 0\end{bmatrix}.$$

By a change-of-basis, we may write the matrix representation of $P_{\text{nullspace}(\mathcal{X})}$ with respect to the standard basis of $\mathbb{R}^{p+1,1}$. Let P denote the change-of-basis matrix $[E_{\text{std}}, \text{id}_{\mathbb{R}^{p+1,1}}, E]$. Then

$$\begin{aligned}
[E_{\text{std}}, P_{\text{nullspace}(\mathcal{X})}, E_{\text{std}}] &= P[E, P_{\text{nullspace}(\mathcal{X})}, E]P^{-1} \\
&= P[E, P_{\text{nullspace}(\mathcal{X})}, E]P^{\top} \\
&= P \begin{bmatrix} I_{(p+1)-r} & 0 \\ 0 & 0 \end{bmatrix} P^{\top} \\
&= P \left(I_{p+1} - \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix} \right) P^{\top} \\
&= PP^{\top} - P \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix} P^{\top} \\
&= I_{p+1} - P \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix} P^{\top} \\
&= I_{p+1} - P_r P_r^{\top}
\end{aligned}$$

where P_r is the matrix that contains the columns $\mathbf{v}_1, \ldots, \mathbf{v}_r$ of P. We can run a similar argument and obtain that the matrix representation of $P_{\text{colspace}(\mathcal{X}^{\top})}$ with respect to the standard basis is $P_r P_r^{\top}$.

Indeed it follows from the spectral decomposition theorem that

$$\text{id}_{\mathbb{R}^{p+1,1}} = P_{\text{nullspace}(\mathcal{X})} + P_{\text{colspace}(\mathcal{X}^{\top})}$$

and so any $\mathbf{w} \in \mathbb{R}^{p+1,1}$ can be written as

$$\begin{aligned}
\mathbf{w} = \text{id}_{\mathbb{R}^{p+1,1}}(\mathbf{w}) &= P_{\text{nullspace}(\mathcal{X})}\mathbf{w} + P_{\text{colspace}(\mathcal{X}^{\top})}\mathbf{w} \\
&= (I_{p+1} - P_r P_r^{\top})\mathbf{w} + P_r P_r^{\top}\mathbf{w}
\end{aligned}$$

The under-braced term varies over all of nullspace($\mathcal{X}$) and so we may re-parameterise $\boldsymbol{\varphi}'$ with it.

Therefore, our general minimum-norm solution to the ordinary least-squares problem, thanks to the singular value decomposition of $\mathcal{X} = Q\Sigma P^{\top}$, is

$$\boldsymbol{\theta} = P\Sigma^{\sim}Q^{\top}\mathbf{y} + (\text{id}_{\mathbb{R}^{p+1,1}} - P_r P_r^{\top})\mathbf{w} \quad \text{for any } \mathbf{w} \in \mathbb{R}^{p+1,1}.$$

6.4 Gram Matrix

Definition 6.4.1 The **Gram matrix** of a collection of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in an inner product space $(V, \langle \cdot, \cdot \rangle)$ is the Hermitian (equal to its own conjugate transpose) matrix of inner products, whose $(i, j)^{\text{th}}$ entry is given by the inner product $\langle v_i, v_j \rangle$.

If the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are the columns of a matrix $\mathcal{X}$, then the Gram matrix is $\mathcal{X}^\dagger \mathcal{X}$ in the general case that the vector coordinates are complex numbers. In the case that the coordinates of the v_i are real numbers, this simplifies to $\mathcal{X}^\top \mathcal{X}$.

Lemma 6.4.2 A real matrix $A \in \mathbb{R}^{n,n}$ is a Gram matrix iff A is symmetric and and all of its eigenvalues are non-negative.

Proof.

$\implies$ Suppose that A is a Gram matrix i.e. $A = \mathcal{X}^T \mathcal{X}$ for some real square matrix $\mathcal{X}$.

$$A^\top = (\mathcal{X}^\top \mathcal{X})^\top = \mathcal{X}^\top (\mathcal{X}^\top)^\top = \mathcal{X}^\top \mathcal{X} = A \qquad \text{so } A \text{ is symmetric.}$$

As for the eigenvalues, let λ be an eigenvalue of A, with eigenvector $\mathbf{v} \neq \mathbf{0}_V$. Then

$$\lambda \|\mathbf{v}\|^2 = \langle A\mathbf{v}, \mathbf{v} \rangle = \langle \mathcal{X}^\top \mathcal{X} \mathbf{v}, \mathbf{v} \rangle = \langle \mathcal{X}\mathbf{v}, \mathcal{X}\mathbf{v} \rangle = \|\mathcal{X}\mathbf{v}\|^2 \geqslant 0$$

so every eigenvalue is non-negative.

$\impliedby$ Denote by E_{std} the standard orthonormal basis for $\mathbb{R}^n$. The associated map $L_A \colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $\mathbf{v} \longmapsto A\mathbf{v}$, where $A = [E_{\text{std}}, L_A, E_{\text{std}}]$ is self-adjoint. By the real spectral theorem, there exists an orthonormal basis $E = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ of V consisting of eigenvectors of L_A (with respective eigenvalues $\lambda_1, \dots, \lambda_n$) s.t. $[E, L_A, E] = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and

$$A = Q[E, L_A, E]Q^\top$$

where Q is an orthogonal matrix whose columns are the $\mathbf{v}_i$. Let $D := \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$. Therefore, $D^2 = [E, L_A, E]$. Now observe that

$$A = QD^2Q^\top = (QD)(DQ^\top) = QD(D^\top Q^\top) = QD(DQ)^\top$$

where we used that D is diagonal and therefore symmetric $D = D^\top$. We conclude that A is a Gram matrix.

■

The normal equations of ordinary linear regression feature a Gram matrix.

$$\mathcal{X}^\top \mathbf{y} = \mathcal{X}^\top \mathcal{X} \hat{\boldsymbol{\theta}}$$

Since $\mathcal{X}$ is real symmetric and positive semidefinite, it's orthogonally diagonalisable i.e. there exists some orthogonal matrix $Q \in \mathbb{R}^{p+1,p+1}$ whose columns are the eigenvectors of $\mathcal{X}^\top \mathcal{X}$ which form an orthonormal basis $(\mathbf{q}_1, \dots, \mathbf{q}_{p+1})$ of $\mathbb{R}^{p+1,1}$ s.t. $\mathcal{X}^\top \mathcal{X} = Q \operatorname{diag}(\lambda_1, \dots, \lambda_{p+1}) Q^\top$. Therefore our normal equations become

$$\mathcal{X}^\top \mathbf{y} = Q \operatorname{diag}(\lambda_1, \dots, \lambda_{p+1}) Q^\top \hat{\boldsymbol{\theta}} \implies Q^\top \hat{\boldsymbol{\theta}} = \operatorname{diag}(1/\lambda_1, \dots, 1/\lambda_{p+1}) Q^{-1} \mathcal{X}^\top \mathbf{y}.$$

By the transformation $\boldsymbol{\alpha} = Q^\top \mathbf{y}$, $\mathbf{c} = Q^{-1} \mathcal{X}^\top \mathbf{y}$, we rotate our normal equations into the eigenbasis of $\mathcal{X}^\top \mathcal{X}$.

$$\boldsymbol{\alpha} = \operatorname{diag}(1/\lambda_1, \dots, 1/\lambda_{p+1}) \mathbf{c}$$

and so the normal equations fully decouple and each rotated coefficient is solved independently as $\alpha_i = c_i/\lambda_i$. If we left-multiply by Q, we recover the least-squares estimate in a slightly different form:

$$\widehat{\boldsymbol{\theta}} = Q\boldsymbol{\alpha} = Q \operatorname{diag}(1/\lambda_1, \dots, 1/\lambda_{p+1}) \mathbf{c} = \sum_{i=1}^{p+1} \frac{c_i}{\lambda_i} \mathbf{q}_i,$$

where $c_i = \mathbf{q}_i^\top \mathcal{X}^\top \mathbf{y}$. Thus, $\widehat{\boldsymbol{\theta}}$ is a weighted sum of the eigenvectors $\mathbf{q}_i$ of $\mathcal{X}^\top \mathcal{X}$. What do these weights represent?

- This is the easier one to reason out because λ_i is an eigenvalue of $\mathcal{X}^\top \mathcal{X}$ i.e.

$$\mathcal{X}^\top \mathcal{X} = Q \operatorname{diag}(1/\lambda_1, \dots, 1/\lambda_{p+1}) Q^\top \implies \operatorname{diag}(1/\lambda_1, \dots, 1/\lambda_{p+1}) = Q^\top \mathcal{X}^\top \mathcal{X} Q$$

and so $\lambda_i = \mathbf{q}_i^\top (\mathcal{X}^\top \mathcal{X}) \mathbf{q}_i = (\mathcal{X}\mathbf{q}_i)^\top \mathcal{X}\mathbf{q}_i = \|\mathcal{X}\mathbf{q}_i\|^2$. Now note that

$$\mathcal{X}\mathbf{q}_i = \begin{bmatrix} — (x^{(1)})^\top — \\ \vdots \\ — (x^{(n)})^\top — \end{bmatrix} \mathbf{q}_i = \begin{bmatrix} (x^{(1)})^\top \mathbf{q}_i \\ \vdots \\ (x^{(n)})^\top \mathbf{q}_i \end{bmatrix} = \begin{bmatrix} \langle \mathbf{q}_i, x^{(1)} \rangle \\ \vdots \\ \langle \mathbf{q}_i, x^{(n)} \rangle \end{bmatrix}$$

is the vector of scalar projections of the training example inputs $x^{(j)}$ onto $\mathbf{q}_i$. Therefore, $\lambda_i = \|\mathcal{X}\mathbf{q}_i\|^2$ measures the total squared magnitude of these projections.

- The numerator c_i can be re-written as

$$c_i = \mathbf{q}_i^\top \mathcal{X}^\top \mathbf{y} = (\mathcal{X}\mathbf{q}_i)^\top \mathbf{y} = \langle \mathbf{y}, \mathcal{X}\mathbf{q}_i \rangle$$

which measures how much the response $\mathbf{y}$ and the data co-vary along $\mathbf{q}_i$.

So each eigenvector $\mathbf{q}_i$ contributes to the linear combination, scaled by how strongly the data project onto that direction (represented by c_i) and inversely proportional to how much the data varies along it (represented by λ_i).

CHAPTER 7

Miscellaneous Topics in Linear Algebra

7.1 Analogy between $\mathrm{Hom}_{\mathbb{K}}(V;V)$ and $\mathbb{C}$

Since $\mathbb{C}$ is a ring under addition $+$ and scalar multiplication $\cdot$, and $\mathrm{Hom}_{\mathbb{K}}(V;V)$ is a ring under addition of linear maps, and composition, we can make some analogies between their elements. Note that $\mathrm{Hom}_{\mathbb{K}}(V;V)$ is not commutative in general (unlike $\mathbb{C}$ which is) and so has a richer theory than $\mathbb{C}$.

- The adjoint on $\mathrm{Hom}_{\mathbb{K}}(V;V)$ plays a similar role to complex conjugation on $\mathbb{C}$.

  $$z \in \mathbb{C} \text{ is real } \iff \overline{z} = z.$$

  Therefore, a self-adjoint operator $T^\dagger = T$ is analogous to a real number.

- The unit circle of $\mathbb{C}$ is the collection of complex numbers z s.t. $\overline{z}z = |z| = 1$. This is analogous to the set of operators on V satisfying $T^\dagger \circ T = \mathrm{id}_V$ i.e. T-unitary.

- Furthermore, non-negative real numbers in $\mathbb{C}$ are analogous to positive semidefinite operators.

A linear operator $T\colon V \to V$ is called positive semidefinite if T is self-adjoint and $\langle T\mathbf{x}, \mathbf{x}\rangle \geqslant 0$ for all $\mathbf{x} \in V$.

7.2 QR Factorisation

The Gram-Schimdt orthogonalisation process is defined by $\mathbf{v}_1 = \mathbf{w}_1$ and

$$\mathbf{v}_k = \mathbf{w}_k - \sum_{j=1}^{k-1} \frac{\langle \mathbf{w}_k, \mathbf{v}_j\rangle}{\|\mathbf{v}_j\|^2}\mathbf{v}_j \qquad \text{for } 2 \leqslant k \leqslant n.$$

If we denote by $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ the orthonormal basis obtained by normalising the orthogonal basis $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, then

$$\mathbf{w}_k = \|\mathbf{v}_k\|\mathbf{u}_k + \sum_{j=1}^{k-1} \langle \mathbf{w}_k, \mathbf{u}_j\rangle \mathbf{u}_j \qquad \text{for } 1 \leqslant k \leqslant n.$$

Theorem 7.2.1 ([2, p. 264]) Suppose that A is a square matrix with linearly independent columns $\mathbf{w}_1, \ldots, \mathbf{w}_n \in \mathbb{K}^{n,1}$. Then there exist unique matrices Q and R s.t. Q is unitary, R is upper triangular with only positive numbers on its diagonal, and $A = QR$.

Proof. Existence: Apply the Gram-Schimdt orthogonalisation process to obtain $\mathbf{v}_1, \ldots, \mathbf{v}_n$ which are orthogonal, and then normalise to obtain an orthonormal basis $E = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ of $\mathbb{K}^{n,1}$ s.t.

$$\mathrm{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k) = \mathrm{span}(\mathbf{e}_1, \ldots, \mathbf{e}_k) \qquad \text{for } k = 1, \ldots, n.$$

Let R be the $n \times n$ matrix whose $(i,j)^{\text{th}}$ entry is $\langle \mathbf{w}_k, \mathbf{e}_j\rangle$. If $j > k$, then $\mathbf{e}_j$ is orthogonal to $\mathrm{span}(\mathbf{e}_1, \ldots, \mathbf{e}_k) = \mathrm{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$ so $\langle \mathbf{w}_k, \mathbf{e}_j\rangle = 0$. Thus, R is upper triangular.

Let Q be the unitary matrix whose columns are $\mathbf{e}_1, \ldots, \mathbf{e}_n$. The k^{th} column of QR is equal to the linear combination of the columns of Q with coefficients coming from the k^{th} column of R i.e.

$$\langle \mathbf{w}_k, \mathbf{e}_1\rangle \mathbf{e}_1 + \ldots + \langle \mathbf{w}_k, \mathbf{e}_k\rangle \mathbf{e}_k$$

which is equal to $\mathbf{w}_k$ because this is the representation of $\mathbf{v}_k$ with respect to the orthonormal basis E for $\mathbb{K}^{n,1}$. Therefore, $A = QR$.

For uniqueness, see Axler. ◧

Bibliography

[1] Stephen Howard Friedberg, Arnold Joseph Insel, and Lawrence Edward Spence. *Linear Algebra.* Prentice Hall, 4th edition, 2003.

[2] Sheldon Jay Axler. *Linear Algebra Done Right.* Springer, 4th edition, 2024.

[3] David Bruce Surowski. *Advanced Linear Algebra.* 1997.