MA231
Vector Analysis
Revision Guide

Written by Khallil Benyattou

WMS the Warwick Mathematics Society

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Introduction

Authors

Written by Khallil Benyattou to reflect the 2016 course taught by Dr. Mario Micallef.
General structure and chronology inspired by D.S. McCormick's 2007 edition.

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History

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Author's Note

I'd like to offer my sincere thanks to Christopher Nguyen for allowing me to view his scanned notes for the module. Without his help with proof-reading, the quality of this guide wouldn't be the same.

1 Review of MA134 Geometry and Motion

1.1 Notation

Notation's always a tricky thing to get right. Sometimes it's too suggestive and at others, too obscure. This happens to be the case with multivariable calculus, most notably derivatives. Indeed, a conference held in 1903 with the aim of agreeing upon standard notation for the partial derivative did the complete opposite of what was intended. Long story short, there are now three more in existence. We aim to be consistent with the notation used by the good doctor Mario Micallef himself.

- We will write $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ for a vector in $\mathbb{R}^n$, and $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ for a function with components $f_1, \ldots, f_m \colon \mathbb{R}^n \to \mathbb{R}$.
- A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called a *scalar field*, and a function $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ is called a *vector field*.
- Given $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$, a curve starting from $\mathbf{p}$ and ending at $\mathbf{q}$ is denoted by $C_{\mathbf{pq}}$.
- For $\Omega \subset \mathbb{R}^n$, $\partial\Omega$ will denote the boundary of Ω, and $\overline{\Omega}$ the closure of Ω.
  Notably, we can observe that $\overline{\Omega} = \Omega \cup \partial\Omega$.

Of course, notation elsewhere will vary but if you understand the general gist of the theorems and results, interpreting them in other notation shouldn't be too much of a problem.

1.2 The Tangential Line Integral

Definition 1.1. The *tangential line integral* of a vector field $\mathbf{v}$ along a curve $C_{\mathbf{pq}}$ that's parameterised by $\mathbf{r}\colon [a,b] \to \mathbb{R}^n$, with $\mathbf{r}(a) = \mathbf{p}$ and $\mathbf{r}(b) = \mathbf{q}$, is defined by

$$\int_{C_{\mathbf{pq}}} \mathbf{v} \cdot \mathrm{d}\mathbf{r} := \int_a^b \mathbf{v}(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\, \mathrm{d}t.$$

Note that this integral depends on the orientation of the path. However, it does not depend on the parametrisation of the curve. In particular, we can parametrise the curve $C_{\mathbf{pq}}$ (of length l) by arc-length s and re-write our tangential line integral as:

$$\int_{C_{\mathbf{pq}}} \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \int_0^l \mathbf{v}(\mathbf{r}(s)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\, \mathrm{d}s.$$

Definition 1.2. For $f\colon \mathbb{R}^n \to \mathbb{R}$, the *gradient* of f is the vector field ∇f defined by

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

Definition 1.3. A *gradient vector field* is a vector field $\mathbf{v}\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathbf{v} = \nabla f$ for some $f\colon \mathbb{R}^n \to \mathbb{R}$. If this is the case, we call f a *scalar potential*[1] of $\mathbf{v}$.

The Chain Rule. Given a function $f\colon \mathbb{R}^n \to \mathbb{R}$ and a path $\mathbf{r}\colon [a,b] \to \mathbb{R}^n$, we can define a new map $h\colon [a,b] \to \mathbb{R}$ to describe how our scalar field acts along the path by $h(t) := f(\mathbf{r}(t))$. Then

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(f(\mathbf{r}(t))\right) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{r}(t)) \frac{\mathrm{d}x_i}{\mathrm{d}t} = \nabla f(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t},$$

where $\mathbf{r}(t) = (x_1(t), \ldots, x_n(t))$ and $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \left(\dfrac{\mathrm{d}x_1}{\mathrm{d}t}, \ldots, \dfrac{\mathrm{d}x_n}{\mathrm{d}t}\right)$.

[1] It's worth mentioning that scalar potentials aren't unique. This is because $\nabla f = \nabla g$ whenever $g(\mathbf{x}) := f(\mathbf{x}) + c$ for every real number c. Although not every vector field is a gradient field, they happen to be quite useful as they allow us to easily evaluate line integrals.

FTC for Gradient Fields. Given a function $f\colon \mathbb{R}^n \to \mathbb{R}$ and a curve $C_{\mathbf{pq}}$ from $\mathbf{p}$ to $\mathbf{q}$ parameterised by $\mathbf{r}\colon [a,b] \to \mathbb{R}^n$, we have that

$$\int_{C_{\mathbf{pq}}} \nabla f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{q}) - f(\mathbf{p}).$$

Definition 1.4. A vector field $\mathbf{v}$ is called *conservative* if for all closed curves $C \subset \mathbb{R}^n$, $\oint_C \mathbf{v} \cdot \mathrm{d}\mathbf{r} = 0$.

Proposition 1.5. A vector field $\mathbf{v}\colon \mathbb{R}^n \to \mathbb{R}^n$ is conservative iff $\forall\ \mathbf{p}, \mathbf{q} \in \mathbb{R}^n$, the tangential line integral

$$\int_{C_{\mathbf{pq}}} \mathbf{v} \cdot \mathrm{d}\mathbf{r}$$

is independent of the choice of path $C_{\mathbf{pq}}$ from $\mathbf{p}$ to $\mathbf{q}$.

Theorem 1.6. A vector field $\mathbf{v}$ is a gradient field iff it's conservative.

1.3 Flux

Definition 1.7. Given $\mathbf{v} = (a,b) \in \mathbb{R}^2$, define $\mathbf{v}^\perp := (b, -a)$. $\mathbf{v}^\perp$ is pronounced $\mathbf{v}$ perp and is the rotation of $\mathbf{v}$ clockwise by 90 degrees.[2]

Definition 1.8. Given a regular curve C that's parameterised by $\mathbf{r}(t) = (x(t), y(t))$,

- a tangent to C is given by $\mathbf{r}'(t) = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\right)$ and
- a normal to C is given by $\mathbf{N}(t) := \mathbf{r}'^\perp(t) = \left(\frac{\mathrm{d}y}{\mathrm{d}t}, -\frac{\mathrm{d}x}{\mathrm{d}t}\right)$.

Definition 1.9. The *flux* of a planar vector field $\mathbf{v}(x,y) = (a(x,y), b(x,y))$ across a curve $C_{\mathbf{pq}}$ parameterised by $\mathbf{r} : [a,b] \to \mathbb{R}^2$ is defined to be the integral

$$\int_C \mathbf{v} \cdot \mathbf{n}\ \mathrm{d}s = \int_a^b \mathbf{v}(\mathbf{r}(t)) \cdot \mathbf{N}(t)\ \mathrm{d}t,$$

where n is the unit normal to C and s is again the arc length parameter of C. The *flux* can be thought of as a measure of how much the field runs across the oriented curve in question. This is as opposed to the field running along the curve, where we discussed the *tangential* line integral.

Definition 1.10 (Coordinate Transformations in $\mathbb{R}^3$)**.** Let f be a map from $\mathbb{R}^3$ to itself. If $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$ is a re-parameterisation of $\Gamma \subset \mathbb{R}^3$ and maps Γ into $\Omega \subset \mathbb{R}^3$, then[3]

$$\iiint_\Omega f\ \mathrm{d}V = \iiint_\Gamma f \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\ \mathrm{d}u\ \mathrm{d}v\ \mathrm{d}w$$

Definition 1.11. Given a vector field $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ and a surface S in $\mathbb{R}^3$ which is parameterised by $\mathbf{r} : \Omega \to \mathbb{R}^3$, the flux of $\mathbf{v}$ through S is given by:

$$\iint_S \mathbf{v} \cdot \mathbf{n}\ \mathrm{d}S = \iint_\Omega (\mathbf{v} \cdot \mathbf{n}) \left\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right\|\ \mathrm{d}A = \pm \iint_\Omega \mathbf{v} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\ \mathrm{d}A$$

where $\mathbf{n}$ is a unit normal to S whose direction $\pm$ will be indicated by the context.

[2] This cannot be done in $\mathbb{R}^n$ for $n \geqslant 3$ as there is no notion of rotation by 90 degrees.

[3] It's understood that for the integral on the right, we substitute for (x,y,z) in terms of (u,v,w) in f.

2 The Integral Theorems of Multivariable Calculus

2.1 Green's Theorem

Definition 2.1. If $\Omega \subset \mathbb{R}^2$ is a region with outward unit normal $\mathbf{n}$, then the *positively oriented unit tangent with respect to* $\mathbf{n}$ is defined as $\mathbf{t} := -\mathbf{n}^{\perp}$.

Definition 2.2. Given a planar vector field $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathbf{v}(x, y) := (a(x, y), b(x, y))$, its curl[4] is a function given by

$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} := \frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}.$$

Green's Theorem. Let $\Omega \subset \mathbb{R}^2$ be a region, $\mathbf{v}(x, y) = (a(x, y), b(x, y))$ be a planar vector field on Ω and $\mathbf{t}$ be the positively oriented unit tangent to the boundary $\partial\Omega$. Then,

$$\iint_{\Omega} \operatorname{curl} \mathbf{v} \, \mathrm{d}A = \oint_{\partial\Omega} \mathbf{v} \cdot \mathbf{t} \, \mathrm{d}s.$$

We've been discussing regions a whole lot so it's good to get a precise feel for what they are.

Definition 2.3. A subset Ω of $\mathbb{R}^n$ is called a region if there's a map $f : \mathbb{R}^n \to \mathbb{R}$ satisfying the following conditions:

(i) All partial derivatives of f are continuous.

(ii) The relation between Ω and f is $\Omega = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) < 0\}$.

(iii) $\nabla f(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in \partial\Omega := \{\mathbf{x} : f(\mathbf{x}) = 0\}$.

If there is such an f, we call it a defining function of the region Ω. Note that f isn't unique i.e. there are many possible choices of defining function.

2.2 Divergence Theorem

Definition 2.4. Let $\mathbf{v}(\mathbf{x}) = (v_1(x_1, \ldots, x_n), \ldots, v_n(x_1, \ldots, x_n))$ be a vector field on $\mathbb{R}^n$. Then, the *divergence* of $\mathbf{v}$ is defined to be:

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} := \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i}$$

Divergence Theorem. Let Ω be a solid region in $\mathbb{R}^3$ and $\mathbf{v} : \Omega \to \mathbb{R}^3$ be a vector field on Ω. Then

$$\iiint_{\Omega} \nabla \cdot \mathbf{v} \, \mathrm{d}V = \iint_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}A$$

where $\mathbf{n}$ is the unit outward normal to Ω.

The Divergence Theorem also applies to planar regions, $\Omega \subset \mathbb{R}^2$. This follows from Green's Theorem applied to $\mathbf{v}^{\perp}$ where $\mathbf{n}$ is an outward pointing unit normal from Ω.

$$\iint_{\Omega} \nabla \cdot \mathbf{v} \, \mathrm{d}A = \oint_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}s$$

If the divergence of a vector field is locally constant in a region, we can say that the divergence is a measure of the flux per unit volume of the vector field.

[4]The reason for the notation $\nabla \times \mathbf{v}$ is because the curl of a vector field $\mathbf{v} = (a, b, c)$ in $\mathbb{R}^3$ is given precisely by the determinant below. This determinant isn't a literal cross product of a "vector of differential operators" and the vector field, but it's reminiscent.

$$\operatorname{curl} \mathbf{v} := \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ a & b & c \end{vmatrix}$$

An interesting case of **2.4** is when we take the divergence of a gradient field $\mathbf{v} = \nabla f$.

Definition 2.5. Let $f : U \to \mathbb{R}$ be a scalar field for some open subset U of $\mathbb{R}^n$. The *Laplacian* of f is defined as

$$\Delta f := \nabla \cdot (\nabla f) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

Definition 2.6. A function $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$ is open, with $\Delta f = 0$ is known as *harmonic.*

Lemma 2.7. Let $f \colon \mathbb{R}^3 \to \mathbb{R}$ be harmonic. Then for all regions $\Omega \subset \mathbb{R}^3$, we have that

$$\iint_{\partial\Omega} \nabla f \cdot \mathbf{n} \, \mathrm{d}A = 0.$$

2.3 Stokes' Theorem

Definition 2.8. A curve or surface is regular if there exists a parameterisation whose derivative is well defined and non-zero at all points. When we say regular, we mean that the object doesn't have any corners or cusps.

Definition 2.9. A surface $S \subset \mathbb{R}^3$ is *regular* if $\forall\, \mathbf{p} \in S$, $\exists\, \epsilon > 0$ and there's a region $\Omega \subset \mathbb{R}^2$ such that $\mathbb{B}(\mathbf{p}, \epsilon, S) = \{\mathbf{x} \in S \colon \|\mathbf{x} - \mathbf{p}\| < \epsilon\}$ can be parameterised by a regular map $\mathbf{r} : \Omega \to \mathbb{R}^3$ i.e. at all points in Ω, we have that

$$\mathbf{n} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \neq 0.$$

So on a regular surface S, we can assign a unit normal vector given by $\mathbf{n}/\|\mathbf{n}\|$ on small pieces of the surface.

Definition 2.10. A regular surface $S \subset \mathbb{R}^3$ is *orientable* if a choice of unit normal can be continuously assigned on all of S.

Stokes' Theorem[5]. Let S be a regular surface in $\mathbb{R}^3$, oriented by a continuous choice of unit normal $\mathbf{n}$, and let $\mathbf{v}$ be a vector field on $\mathbb{R}^3$. Let $\mathbf{t}$ be a unit tangent to ∂S which is positively oriented with respect to $\mathbf{n}$ in the sense that $\mathbf{n} \times \mathbf{t}$ points into S. Then,

$$\iint_S \operatorname{curl} \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}A = \int_{\partial S} \mathbf{v} \cdot \mathbf{t} \, \mathrm{d}s.$$

2.4 Applications

Lemma 2.11. Let Ω be a solid region in $\mathbb{R}^3$ and $f \colon \Omega \to \mathbb{R}^3$ be continuously differentiable on Ω. Then[6] $\nabla \times (\nabla f) = 0$ in Ω.

Proof. Let S be an arbitrary surface. Applying Stokes' Theorem to ∇f on S gives

$$\iint_S \nabla \times (\nabla f) \cdot \mathbf{n} \, \mathrm{d}A = \int_{\partial S} \nabla f \cdot \mathbf{t} \, \mathrm{d}s \overset{\text{FTC}}{=} f(\mathbf{q}) - f(\mathbf{p}) = 0$$

□

Lemma 2.12. Let Ω be a solid region in $\mathbb{R}^3$ and $\mathbf{v} : \Omega \to \mathbb{R}^3$ be a vector field. Then $\nabla \cdot (\nabla \times \mathbf{v}) = 0$.

Proof.

$$\begin{aligned} \iiint_\Omega \nabla \cdot (\nabla \times \mathbf{v}) \, \mathrm{d}V &= \iint_{\partial\Omega} \nabla \times \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}A \quad \text{by the Divergence Theorem} \\ &= \int_{\partial(\partial S)} \mathbf{v} \cdot \mathbf{t} \, \mathrm{d}s \quad \text{via Stokes' Theorem} \\ &= 0 \quad \text{since } \partial\,(\partial S) = \varnothing. \end{aligned}$$

As Ω is arbitrary, we're forced to conclude that $\nabla \cdot (\nabla \times \mathbf{v}) = 0$. □

[5]This is Green's Theorem for orientable surfaces with boundary in $\mathbb{R}^3$.

[6]Take note that the calculus identity $\nabla \times (\nabla \cdot) = 0$ for some function is closely related to the topological fact that $\partial\,(\partial \cdot) = \varnothing$ for some surface.

Definition 2.13. A region $\Omega \subset \mathbb{R}^3$ is said to have *trivial* 1^{st} *homology* if, inside of Ω, every closed curve C is the boundary of some surface S.

> **Proposition 2.14.** If $\Omega \subset \mathbb{R}^3$ has trivial 1^{st} homology and $\mathbf{v}\colon \Omega \to \mathbb{R}^3$ satisfies $\nabla \times \mathbf{v} = 0$, then $\mathbf{v}$ is conservative and admits a scalar potential.

Definition 2.15. A region $\Omega \subset \mathbb{R}^3$ is said to have *trivial* 2^{nd} *homology* if, inside of Ω, every closed surface S is the boundary of some region E.

> **Proposition 2.16.** If $\Omega \subset \mathbb{R}^3$ has trivial 2^{nd} homology and $\mathbf{v}\colon \Omega \to \mathbb{R}^3$ satisfies $\nabla \cdot \mathbf{v} = 0$, then $\mathbf{v}$ admits a vector potential i.e. $\exists\, \mathbf{w}\colon \Omega \to \mathbb{R}^3$ such that $\mathbf{v} = \nabla \times \mathbf{w}$.

3 Introduction to Complex Analysis

Our next aim is to understand the calculus of functions $f\colon \mathbb{C} \to \mathbb{C}$. Writing $f(x+iy) = u(x,y) + iv(x,y)$ where $u, v\colon \mathbb{R}^2 \to \mathbb{R}$, we see that there is a one-to-one correspondence between functions $f\colon \mathbb{C} \to \mathbb{C}$ and vector fields of the form $(u,v)\colon \mathbb{R}^2 \to \mathbb{R}^2$. We'll capitalise upon this link to apply the Divergence Theorem and Stokes' Theorem to derive very useful results.

3.1 Continuity & Convergence

The definitions of convergence and continuity for complex functions are analogous to their real counterparts. However, as one would expect, $\mathbb{R} \subset \mathbb{C}$ so considering intervals no longer suffices. Instead, we consider balls:

Definition 3.1. For $z \in \mathbb{C}$ and $\varepsilon > 0$, the *open ball of radius* ε *around* z is $\mathbb{B}(z,\varepsilon) = \{w \in \mathbb{C} : |w - z| < \varepsilon\}$.

Definition 3.2. A set $E \subset \mathbb{C}$ is *open* if $\forall z \in E$, $\exists\, r > 0$ such that $\mathbb{B}(z,r) \subset E$.

From now on, E will always denote an open subset of $\mathbb{C}$.

Definition 3.3. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ where $z_n = x_n + iy_n$. Then (z_n) *converges* to some $z \in \mathbb{C}$ if $|z_n - z| \to 0$ as $n \to \infty$. More precisely, this means that $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that whenever $n \geqslant N$, $z_n \in B(z,\varepsilon)$. As usual, z is called the limit of the sequence and we write that $z_n \to z$.

The usual rules for limits of sequences follow from this definition. If $(z_n) \to z$ and $(w_n) \to w$ then $(z_n + w_n) \to z + w$, $(z_n w_n) \to zw$ and $z_n/w_n \to z/w$ so long as $w \neq 0$.

Definition 3.4. For $E \subset \mathbb{C}$, $f\colon E \to \mathbb{C}$ is *continuous* at $z \in E$ if $z_n \to z \implies f(z_n) \to f(z)$.
Equivalently, f is continuous at z if $\forall \varepsilon > 0$, $\exists\, \delta_z > 0$ such that $w \in B(z, \delta_z) \implies f(w) \in B(f(z), \varepsilon)$.

Again, the usual rules for continuous functions follow. If $f, g : E \to \mathbb{C}$ are continuous at z, then so are $f+g$, fg and f/g. For the last case, we of course require that $g(z) \neq 0$. We're now ready to define differentiability in the complex plane.

3.2 Complex Differentiation

Definition 3.5. For $E \subset \mathbb{C}$, $f\colon E \to \mathbb{C}$ is *complex differentiable*[7] at $z \in E$ if the following limit exists (and is finite)

$$\lim_{\substack{h\to 0\\ \text{in } \mathbb{C}}} \frac{f(z+h) - f(z)}{h}.$$

If the limit does indeed exist, we call the limiting value the *derivative* of f at z, written $f'(z)$.

Note that this condition is more restrictive than in the real case. This is because there are more ways for a sequence (h_n) to tend to 0 in $\mathbb{C}$. The following examples serve to illustrate this point more concretely.

[7]This is equivalent to saying that the limit $\lim_{n\to\infty} \dfrac{f(z+h_n) - f(z)}{h_n}$ exists for *every* null sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$.

Example 3.6. Let $f\colon \mathbb{C} \to \mathbb{C}$ defined by:

1. $f(z) = z$ is differentiable at every point of $\mathbb{C}$, and $f'(z) = 1$.
2. $f(z) = \overline{z}$ is *not* differentiable anywhere in the complex plane. Consider the difference quotient

$$\delta := \frac{f(z+h_n) - f(z)}{h_n} = \frac{\overline{z+h_n} - \overline{z}}{h_n} = \frac{\overline{h_n}}{h_n}.$$

In the case that $h_n = 1/n$, $\delta = 1$. If $h_n = i/n$, then $\delta = -i/i = -1$. Since these two differ, the limit doesn't exist and f is accordingly not differentiable at any point $z \in \mathbb{C}$.

I'm beginning to sound like a parrot but the usual rules for differentiation i.e. the product, quotient and chain rules remain valid for complex differentiation.

3.3 Interpretations of Complex Valued Functions

There are three ways to view complex valued functions:

(i) In the naïve algebraic way that we have been.

(ii) As functions $F : \mathbb{R}^2 \to \mathbb{R}^2$ where we make the following correspondences:

- $z = x + iy \in \mathbb{C} \longleftrightarrow (x, y) \in \mathbb{R}^2$
- $f(z) = u(z) + iv(z) \in \mathbb{C} \longleftrightarrow F(x, y) = (u(x, y), v(x, y)) \in \mathbb{R}^2$

(iii) $f\colon E \to \mathbb{C}$ may be viewed as a vector field $\mathbf{f} : E \to \mathbb{R}^2$. If $f = u + iv$, then $\mathbf{f}$ is the vector field[8] $\mathbf{f}(x, y) = (u(x, y), -v(x, y))$.

Using our correspondence, we'll look at complex and real differentiability. Let the complex number $h = h_1 + ih_2$ correspond to the real vector $\mathbf{h} = (h_1, h_2)$.

Definition 3.7. A function $F : E \to \mathbb{R}^2$ is Fréchet differentiable at $(x_0, y_0) \in E$ if there is a bounded linear map $A_{(x_0,y_0)} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{\|\mathbf{h}\|\to 0} \frac{\|F(x_0 + h_1, y_0 + h_2) - F(x_0, y_0) - A_{(x_0,y_0)}\mathbf{h}\|}{\|\mathbf{h}\|} = 0.$$

The linear map $A_{(x_0,y_0)}$ is then called the Fréchet derivative of F at (x_0, y_0). Using the standard basis of $\mathbb{R}^2$, our linear map $A_{(x_0,y_0)}$ has a matrix representation. In fact, it's the familiar Jacobian matrix!

$$A_{(x_0,y_0)} = \begin{bmatrix} \dfrac{\partial u}{\partial x}(x_0, y_0) & \dfrac{\partial u}{\partial y}(x_0, y_0) \\ \dfrac{\partial v}{\partial x}(x_0, y_0) & \dfrac{\partial v}{\partial y}(x_0, y_0) \end{bmatrix} = \mathrm{DF}(x_0, y_0)$$

Given $f = u + iv$, we can say that f is continuous if and only if u and v are both continuous. However, the same is not true of complex differentiation. As a counterexample, consider $f(z) = \overline{z}$ so that $f(x + iy) = x - iy$. Letting $u(x, y) = x$ and $v(x, y) = -y$, we see that both u and v are differentiable but f is most certainly not complex differentiable according to **3.6**. Naturally, as mathematicians we want to see what's up; we seek conditions on u and v that guarantee the differentiability of f!

The thought process is to begin by supposing that f is differentiable and consider the following difference quotients, incrementing f in the directions of the coordinate axes. This means that we'd be looking at $h = \varepsilon$ and $h = \varepsilon i$ as $\varepsilon \to 0$ in $\mathbb{R}$ for the x and y axes respectively.

[8] The significance of the negative sign for the second entry of the vector field is explained in section **3.5**.

The complex differentiability of f tells us that the following two limits are equal (and in fact all limits as $h \to 0$ in $\mathbb{C}$ are too).

$$f'(z) = \lim_{\varepsilon \to 0} \frac{f(z+\varepsilon) - f(z)}{\varepsilon} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \lim_{\varepsilon \to 0} \frac{f(z+\varepsilon i) - f(z)}{\varepsilon i} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Equating real and imaginary parts, we arrive at the *Cauchy-Riemann equations*: $u_x = v_y$ and $v_x = -u_y$. This set of equations is a necessary condition for the complex differentiability of f. If we add continuity of the partial derivatives, we get that it is also a sufficient condition:

Theorem 3.8. A function $f\colon E \to \mathbb{C}$ is complex differentiable at $z_0 = x_0 + y_0 \in E$ if and only if $F\colon E \to \mathbb{R}^2$, defined by $F(x+iy) = (u(x,y), v(x,y))$, is Fréchet differentiable as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ and $\mathrm{DF}(x_0, y_0)$ is of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where $a, b \in \mathbb{R}$. In particular, f satisfies the Cauchy–Riemann equations and

$$f'(z_0) = a + ib = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial u}{\partial y}(x_0, y_0).$$

Definition 3.9. A map $f\colon E \to \mathbb{C}$ is *holomorphic* on E if it is complex differentiable at all points of E.

Holomorphicity requires that the set over which f is $\mathbb{C}$-differentiable is open.

3.4 Complex Line Integrals

Having studied the differentiation of complex functions, we now rear our heads toward integration.

Definition 3.10. Consider an oriented curve C in the complex plane, parameterised by $z : [a,b] \to \mathbb{C}$ defined by $z(t) = x(t) + iy(t)$. Given a continuous function $f\colon \mathbb{C} \to \mathbb{C}$, we define the following integrals:

(i) $\displaystyle\int_C f \, \mathrm{d}z := \int_a^b f(z(t)) \, \frac{\mathrm{d}z}{\mathrm{d}t} \, \mathrm{d}t$ where $\displaystyle\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} + i\frac{\mathrm{d}y}{\mathrm{d}t}$

(ii) $\displaystyle\int_C f \, \mathrm{d}\bar{z} := \int_a^b f(z(t))\frac{\mathrm{d}\bar{z}}{\mathrm{d}t} \, \mathrm{d}t$ where $\displaystyle\frac{\mathrm{d}\bar{z}}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} - i\frac{\mathrm{d}y}{\mathrm{d}t}$

(iii) $\displaystyle\int_C f \, |\mathrm{d}z| := \int_a^b f(z(t)) \left|\frac{\mathrm{d}z}{\mathrm{d}t}\right| \, \mathrm{d}t$ where $\displaystyle\left|\frac{\mathrm{d}z}{\mathrm{d}t}\right| = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}$

Example 3.11. Let $f(z) = 1/z$ where $z \in \mathbb{C} \setminus \{0\}$ and C be the unit circle centred at the origin traversed counterclockwise. We can parameterise C with $z\colon [0, 2\pi] \to \mathbb{C}$ defined by $z(t) = e^{it}$. By invoking the parameterisation at $(*)$, we have that

$$\oint_{|z|=1} \frac{1}{z} \, \mathrm{d}z \overset{(*)}{=} \oint_0^{2\pi} \frac{1}{e^{it}} \, ie^{it} \, \mathrm{d}t = 2\pi i.$$

Upon re-arranging, we arrive at an important identity; a complicated way of getting 1.

$$\frac{1}{2\pi i} \oint_{|z|=1} \frac{1}{z} \, \mathrm{d}z = 1$$

3.5 Linking Complex Analysis and Vector Analysis

Recall that to $f = u + iv$, we associate the vector field $\mathbf{f} = (u, -v)$. Now we'll explain the meaning behind the negative sign on v in $\mathbf{f}$. For a curve C, parameterised by $t \mapsto z(t) = x(t) + iy(t)$, we can expand the following integral and separate it into its real and imaginary parts.

$$\begin{aligned}
\int_C f \ \mathrm{d}z &= \int_a^b \Big(u(x(t), y(t)) + iv(x(t), y(t))\Big)\left(\frac{\mathrm{d}x}{\mathrm{d}t} + i\frac{\mathrm{d}y}{\mathrm{d}t}\right) \mathrm{d}t \\
&= \int_a^b \left(u(x(t), y(t))\frac{\mathrm{d}x}{\mathrm{d}t} - v(x(t), y(t))\frac{\mathrm{d}y}{\mathrm{d}t}\right) \mathrm{d}t + i\int_a^b \left(u(x(t), y(t))\frac{\mathrm{d}y}{\mathrm{d}t} + v(x(t), y(t))\frac{\mathrm{d}x}{\mathrm{d}t}\right) \mathrm{d}t \\
&= \int_a^b (u, -v) \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\right) \mathrm{d}t + i\int_a^b (u, -v) \cdot \left(\frac{\mathrm{d}y}{\mathrm{d}t}, -\frac{\mathrm{d}x}{\mathrm{d}t}\right) \mathrm{d}t \\
&= \int_C \mathbf{f} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \ \mathrm{d}t + i\int_C \mathbf{f} \cdot \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)^{\perp} \mathrm{d}t \\
&= \int_C \mathbf{f} \cdot \mathrm{d}\mathbf{r} + i\int_C \mathbf{f} \cdot \mathbf{n} \ \mathrm{d}s \\
&= (\text{Tangential line integral of } \mathbf{f} \text{ along } C) + i\,(\text{Flux of } \mathbf{f} \text{ across } C)
\end{aligned}$$

It happens to be the case that the Cauchy-Riemann equations are linked to the notions of divergence and curl from the first half of the course. This is a consequence of basic re-arrangement!

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \iff \frac{\partial}{\partial x}(u) + \frac{\partial}{\partial y}(-v) = 0 \iff \nabla \cdot \mathbf{f} = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \iff \frac{\partial}{\partial x}(-v) + \frac{\partial}{\partial y}(-u) = 0 \iff \operatorname{curl} \mathbf{f} = 0$$

Definition 3.12. A vector field $\mathbf{f}$ that satisfies $\operatorname{curl} \mathbf{f} = 0$ is called *irrotational.* If $\mathbf{f}$ satisfies $\nabla \cdot \mathbf{f} = 0$, then it is called *incompressible.*

Proposition 3.13. If $f = u + iv$ satisfies the Cauchy-Riemann equations, then u and v are both harmonic.

Proof.

$$\Delta u = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \overset{\text{CR}}{=} \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = 0$$

□

Cauchy's Theorem. If $f : E \to \mathbb{C}$ is holomorphic and $C \subset E$ is a closed curve that is the boundary of a region $\Omega \subset E$, then

$$\int_C f \ \mathrm{d}z = 0.$$

Proof. As in the beginning of this section, we can write

$$\begin{aligned}
\int_C f \ \mathrm{d}z &= \int_{\partial\Omega} \mathbf{f} \cdot \mathrm{d}\mathbf{r} + i\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} \ \mathrm{d}s \\
&= \underbrace{\iint_\Omega \operatorname{curl} \mathbf{f} \ \mathrm{d}A}_{\text{Green's Thm.}} + i \underbrace{\iint_\Omega \nabla \cdot \mathbf{f} \ \mathrm{d}A}_{\text{Div. Thm.}} \\
&= 0 \text{ by the Cauchy-Riemann equations.}
\end{aligned}$$

□

It's important that our region Ω be simply connected i.e. that it doesn't contain any holes. Otherwise, Cauchy's Theorem doesn't apply. Below is an example on $\mathbb{C} \setminus \{0\}$.

Example 3.14. Recall the integral identity from example **3.11**. This doesn't contradict Cauchy's Theorem because $\{z : |z| = 1\}$ isn't the boundary of any region Ω in $E := \mathbb{C} \setminus \{0\}$. If $\{z : |z| = 1\}$ were the boundary of some region, the region would need to contain the origin.

Estimation Lemma. If we have a continuous function $f : E \to \mathbb{C}$ and a curve $C \subset E$, then

$$\left| \int_C f \, \mathrm{d}z \right| \leqslant \int_C |f| \, |\mathrm{d}z| \, .$$

The estimation lemma will come in handy in the final exam. Make sure to apply it very carefully, only when comparing the magnitudes (absolute values) of complex numbers. Make certain that you're well acquainted with the standard and reverse triangle inequalities. They'll be of paramount importance when trying to bound integrands from above when trying to evaluate integrals.

Evaluation Lemma. If $f\colon E \to \mathbb{C}$ is continuous at $z_0 \in E$, then

$$f(z_0) = \lim_{r \to 0} \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z - z_0)} \, \mathrm{d}z.$$

Cauchy's Theorem concerns itself with a region Ω in which f is holomorphic at every point. Does this remain the case if f isn't holomorphic at a single point in Ω? How about several points? To answer these questions, we must add a few more conditions and this leads us nicely into the discussion of Cauchy's integral formulae.

Cauchy's Integral Formula. Suppose that $f\colon E \to \mathbb{C}$ is holomorphic and $z_0 \in E$. Choose a $\rho > 0$ such that $\overline{\mathbb{B}(z_0, \rho)} \subset E$. Let $C \subset E$ be a closed curve such that there exists a region $\Omega \subset E \setminus \{z_0\}$ with $\partial\Omega = C \cup (-\partial\mathbb{B}(z_0, \rho))$. Then for every $z_0 \in \mathbb{B}(z_0, \rho)$, with the contour integral being taken counterclockwise, we have that

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)} \, \mathrm{d}z.$$

Cauchy's integral formula expresses the fact that a holomorphic function on a closed disk is completely determined by its values on the disk's boundary.

Taylor's Theorem. Let $f\colon E \to \mathbb{C}$ be holomorphic and suppose that $\overline{\mathbb{B}(z_0, R)} \subset E$. Then there exists a sequence of complex numbers a_n such that for all $z \in \mathbb{B}(z_0, R)$, the power series below converges and is equal to $f(z)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

For each $n \in \mathbb{N}_0$, the a_n's are defined by the n^{th} Cauchy Integral Formula (abbreviated as CI_n) given by

$$a_n := \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z.$$

In particular, f is infinitely often $\mathbb{C}$-differentiable on $\mathbb{B}(z_0, R)$ and $a_n = \dfrac{f^{(n)}(z_0)}{n!}$.

3.6 Power Series for Holomorphic Functions

Example 3.15. As an extension of the power series you've studied in your first year analysis courses, the following functions $\mathbb{C} \to \mathbb{C}$ also have their own power series representations:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \text{ where } |z| < 1 \qquad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \qquad \cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \qquad \cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

Example 3.16. The objective is to find the power series expansion of $1+e^{2z}$ about $z_0 = \pi i/2$. Accordingly, we want to find a sequence of terms a_n that puts our original function $1 + e^{2z}$ into the form

$$\sum_{n=0}^{\infty} a_n \left(z - \frac{\pi i}{2}\right)^n.$$

In order to do this, we need to re-arrange our term involving the exponential to align it about $\pi i/2$.

$$1 + e^{2z} = 1 + e^{2\left(z - \frac{\pi i}{2}\right) + \pi i} = 1 - e^{2\left(z - \frac{\pi i}{2}\right)} = 1 - \sum_{n=0}^{\infty} \frac{2^n \left(z - \frac{\pi i}{2}\right)^n}{n!} = -\sum_{n=1}^{\infty} \frac{2^n \left(z - \frac{\pi i}{2}\right)^n}{n!}$$

3.7 Applications of Taylor's and Cauchy's Theorems

Liouville's Theorem. Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded. Then f is constant.

Proof. For any $z \in \mathbb{C}$ and for any $R > 0$, the 1st Cauchy Integral Formula gives

$$f'(z) = \frac{1}{2\pi i} \int_{|w-z|=R} \frac{f(w)}{(w-z)^2}\, \mathrm{d}w.$$

Our second condition is that f is bounded i.e. there exists an $M > 0$ satisfying $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$. By the Estimation Lemma, we have that

$$|f'(z)| \leqslant \frac{M}{2\pi R^2} \int_{|w-z|=R} |\mathrm{d}z| = \frac{M}{R}.$$

Letting $R \to \infty$ tells us that $f'(z) = 0$ for all $z \in \mathbb{C}$ so f is constant. □

Fundamental Theorem of Algebra. Every non-constant polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n \in \mathbb{C}[z]$ has at least one root.

Proof. Suppose that $p(z)$ is non-constant and has no roots. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be entire if it is complex differentiable at every point of $\mathbb{C}$. Then $1/p$ is an entire function and we need only prove that it is bounded and thus constant. If $a_n \neq 0$ for $n \geqslant 1$, $z \neq 0$, we can write

$$\frac{p(z)}{z^n} = \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z}\right) + a_n.$$

Note that the first term converges to 0 as $z \to \infty$ and for sufficiently large $|z|$, $\left|\frac{p(z)}{z^n}\right| \geqslant \frac{|a_n|}{2}$ and

$$\left|\frac{1}{p(z)}\right| = \frac{|z|^n}{|p(z)|} \frac{1}{|z|^n} \leqslant \frac{2}{|a_n|} \frac{1}{|z|^n} \overset{z\to\infty}{\longrightarrow} 0$$

Finally, $1/p$ is bounded and, by Liouville's theorem, constant i.e. $1/p = c \iff p = 1/c$ which contradicts our original assumption that p is non-constant. Hence, p has at least one root. □

Closing Remarks

Vector Analysis isn't a very proof oriented course. Instead, it would benefit you to have a working knowledge of the theorems and be able to make different types of calculations quickly. The exam itself is quite time-pressured so although it's easier said than done, try to keep a calm mind[9] and not rush. Haste makes waste.

Good luck,
Khallil

[9] TM04 for all you Pokémon fans.